

Vafa-Witten Invariants for Projective Surfaces II: Semistable Case

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ABSTRACT. We study extensions of Vafa-Witten invariants on polarised surfaces to the setting where there are semistable Higgs pairs.

We propose a definition using virtual localisation inspired by the pairs of Mochizuki and Joyce-Song, and show it gives the right answer under certain conditions.

We expect for $K_S \leq 0$ (and prove for $\deg K_S < 0$) that our definition coincides with an alternative definition using Behrend localisation, counting semistable Higgs pairs by weighted Euler characteristic.

For K3 surfaces we calculate these invariants assuming a conjecture of Toda. The resulting modular forms agree with and generalise conjectures of Vafa and Witten. We also prove a part of Toda's conjecture in rank 2 by showing the Behrend weighting is 1.

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1. INTRODUCTION

On a polarised surface $(S, \mathcal{O}_S(1))$, solutions of the $U(r)$ Vafa-Witten equations correspond to slope polystable Higgs pairs

$$(E, \phi), \quad \phi \in \text{Hom}(E, E \otimes K_S),$$

where E is a coherent sheaf of rank r on S . To partially compactify the moduli space we use Gieseker semistable Higgs sheaves; see Section 2.1 for definitions and [TT1, Introduction] for a much more detailed account.

Via the spectral construction, Gieseker (semi)stable Higgs pairs (E, ϕ) are equivalent to compactly supported Gieseker (semi)stable torsion sheaves

$$\mathcal{E}_\phi \quad \text{on} \quad X = K_S.$$

Letting $\pi: X = K_S \rightarrow S$ be the projection, the relation between the two sets of data is that

$$(E, \phi) = \left(\pi_* \mathcal{E}_\phi, \pi_*(\eta \cdot \text{id}_{\mathcal{E}_\phi}) \right),$$

where η is the tautological section of $\pi^* K_S$ on $X = K_S$.

Stable case. Fixing the Chern classes $r, c_1, c_2 \in H^*(S)$ of E on S — which is equivalent to fixing the topological type of \mathcal{E}_ϕ on X — there is a quasi-projective moduli space \mathcal{N}_{r,c_1,c_2} of Gieseker semistable Higgs pairs. It is noncompact, but the obvious \mathbb{C}^* action (scaling ϕ , or equivalently acting on the moduli space of torsion sheaves on X by scaling K_S) has projective \mathbb{C}^* -fixed locus. When the Chern classes are chosen so that stability and semistability coincide (for instance if the rank and degree of E are coprime) there is a symmetric obstruction theory [TT1] and we can define a $U(r)$ Vafa-Witten invariant by virtual localisation [GP]. It is just a local DT invariant of X counting the stable torsion sheaves \mathcal{E}_ϕ .

However, this invariant vanishes unless $H^{0,1}(S) = 0 = H^{0,2}(S)$. It is much more interesting to consider an analogue of $SU(r)$ Vafa-Witten theory by picking a line bundle L on S and fixing¹

$$\det E = L \quad \text{and} \quad \text{tr } \phi = 0$$

on S . On X this amounts to fixing the centre of mass of the support of \mathcal{E}_ϕ on each fibre of $\pi: X = K_S \rightarrow S$ to be 0, and $\det \pi_* \mathcal{E}_\phi = L$.

In [TT1] it is shown that the resulting moduli space $\mathcal{N}_{r,L,c_2}^\perp$ also carries a symmetric obstruction theory, so we can define an $SU(r)$ Vafa-Witten invariant by virtual localisation to the compact \mathbb{C}^* -fixed locus,

$$(1.1) \quad \text{VW}_{r,L,c_2} := \int_{[(\mathcal{N}_{r,L,c_2}^\perp)^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \in \mathbb{Q}.$$

¹ $SU(r)$ is a bit of a misnomer unless $L = \mathcal{O}_S$, but by allowing more general L — such as one for which $\deg(E)$ is coprime to $\text{rank}(E)$ — we can ensure semistability implies stability in [TT1]. For the invariants defined in this paper this problem does not arise; we can take $L = \mathcal{O}_S$.

This defines deformation invariant rational numbers whose generating series are expected to give modular forms.

In this paper we describe an extension of this theory in the presence of *strictly semistable Higgs pairs*. It is currently conjectural, though we do show it works in various settings, and recovers \mathbb{VW}_{r,L,c_2} (1.1) when semistability implies stability. First, though, we take a detour through a related invariant.

Behrend localisation. Instead of virtual localisation, one could also consider defining invariants using Behrend localisation.

Let \mathcal{N} be a scheme with a symmetric perfect obstruction theory, such as the moduli space of stable Higgs pairs (E, ϕ) on S , or the moduli space of stable $SU(r)$ Higgs pairs with $\det E = \mathcal{O}_S$ and $\mathrm{tr} \phi = 0$. Behrend [Be] defines a constructible function Kai,

$$\chi^B : \mathcal{N} \longrightarrow \mathbb{Z},$$

such that if \mathcal{N} is compact then the degree of its (zero dimensional) virtual cycle equals the Euler characteristic of \mathcal{N} weighted by χ^B ,

$$(1.2) \quad \int_{[\mathcal{N}]^{\mathrm{vir}}} 1 = e(\mathcal{N}, \chi^B) := \sum_{i \in \mathbb{Z}} i \cdot e((\chi^B)^{-1}(i)).$$

Since our moduli spaces \mathcal{N} are noncompact, we instead take the right hand side as a definition. When \mathcal{N} carries a \mathbb{C}^* action, χ^B is \mathbb{C}^* -invariant. Since the Euler characteristic of any non-fixed orbit is 0, only the fixed points contribute to (1.2), giving the localisation

$$(1.3) \quad e(\mathcal{N}, \chi^B) = e(\mathcal{N}^{\mathbb{C}^*}, \chi^B|_{\mathcal{N}^{\mathbb{C}^*}}).$$

This suggests an alternative definition of a $U(r)$ Vafa-Witten invariant²

$$(1.4) \quad \widetilde{\mathbb{VW}}_{r,c_1,c_2}(S) := e\left(\mathcal{N}_{r,c_1,c_2}^{\mathbb{C}^*}, \chi_{\mathcal{N}_{r,L,c_2}}^B|_{\mathcal{N}_{r,L,c_2}^{\mathbb{C}^*}}\right) \in \mathbb{Z},$$

and an $SU(r)$ Vafa-Witten invariant

$$(1.5) \quad \mathbb{vw}_{r,L,c_2}(S) := e\left((\mathcal{N}_{r,L,c_2}^\perp)^{\mathbb{C}^*}, \chi_{\mathcal{N}_{r,L,c_2}^\perp}^B|_{(\mathcal{N}_{r,L,c_2}^\perp)^{\mathbb{C}^*}}\right) \in \mathbb{Z}.$$

These need not be deformation invariant, and we will see that in general they give the “wrong” definition from the point of view of physics. But in Section 7 we show that $\mathbb{vw} = \mathbb{VW}$ (1.1) when $\deg K_S < 0$, and explain why we think the same should be true when $K_S = \mathcal{O}_S$.

One great advantage of (1.5) over the virtual localisation definition (1.1) of [TT1] is that it immediately extends to the setting where there are strictly semistable Higgs pairs, as we describe next.

²One can also use Kiem-Li’s cosection localisation [KL1] applied to the cosection given by the vector field on \mathcal{N} induced by the \mathbb{C}^* action. The result is the same [Ji, JT].

1.1. Joyce-Song/Kontsevich-Soibelman. Since the work of Joyce-Song [JS] and Kontsevich-Soibelman [KS] uses the Behrend function (and refinements thereof) we can apply their results to the Kai localisation (1.5). In particular this allows us to extend the definition of the integers $\widetilde{\text{vw}}$ (1.4) to allow *strictly semistable Higgs pairs* at the expense of getting $U(r)$ Vafa-Witten invariants

$$\widetilde{\text{vw}}_{r,c_1,c_2}(S) \in \mathbb{Q}$$

which are *rational numbers*. We explain the definition in Section 3. When $H^1(\mathcal{O}_S) \neq 0$ the invariant vanishes, so in Section 4 we modify Joyce-Song's theory to define an $SU(r)$ Vafa-Witten invariant generalising (1.5),

$$\text{vw}_{r,L,c_2}(S) \in \mathbb{Q}$$

“counting” *semistable* Higgs pairs (E, ϕ) with fixed determinant $\det E = L$ and $\text{tr } \phi = 0$. In particular, we can take $L = \mathcal{O}_S$, in which case we often drop L from the notation, denoting the resulting Vafa-Witten invariant by

$$\text{vw}_{r,c_2}(S) \in \mathbb{Q}.$$

While we think this is the “wrong” definition of the Vafa-Witten invariant in general, we expect it to be equal to the “right” one VW (1.1) when $K_S \leq 0$. Therefore it makes sense to study it on K3 surfaces.

1.2. K3 surfaces. Using a wonderful conjecture of Toda [To1] — which we prove part of in rank $r = 2$ — we compute the generating series of invariants vw on a K3 surface S in Section 5.

For r, c_2 coprime a vanishing theorem holds, as exploited in [VW], and we need only consider the Euler characteristics of smooth moduli spaces of stable sheaves on S . These have a modular generating series.

But when r and c_2 are not coprime, there exist strictly semistable sheaves and nontrivial Higgs pairs with $\phi \neq 0$. In (5.25) we calculate the generating series to be

$$(1.6) \quad \sum_{c_2} \text{vw}_{r,c_2} q^{c_2} = \sum_{d|r} \frac{d}{r^2} q^r \sum_{j=0}^{d-1} \eta \left(e^{\frac{2\pi i j}{d}} q^{\frac{r}{d^2}} \right)^{-24}.$$

When r is prime, so that d takes only the values 1 and r , this recovers a prediction of [VW, End of Section 4.1],

$$-\frac{1}{r^2} q^r \eta(q^r)^{-24} - \frac{1}{r} q^r \sum_{j=0}^{r-1} \eta \left(e^{\frac{2\pi i j}{r}} q^{1/r} \right)^{-24}.$$

Vafa and Witten also asked for the extension to more general r , which is what (1.6) gives.

1.3. Virtual localisation and semistables. In Section 7 we develop something like the Joyce-Song pairs formalism, but using virtual localisation instead of Behrend localisation. Somewhat to our surprise, with a small modification of the formulae this seems to work well to define VW in the presence of strictly semistable Higgs pairs.

That is, motivated by Mochizuki's pairs [Mo, Section 7.3.1] and Joyce-Song pairs [JS], we virtually enumerate certain stable pairs

$$(\mathcal{E}_\phi, s)$$

on $X = K_S$ (or equivalently stable triples (E, ϕ, s) on S .) Here the torsion sheaf \mathcal{E}_ϕ has centre of mass zero on the K_S fibres (equivalently $\text{tr } \phi = 0$), $\det \pi_* \mathcal{E}_\phi = \det E \cong L$, and

$$s \in H^0(X, \mathcal{E}_\phi(n)) \cong H^0(S, E(n))$$

for some fixed $n \gg 0$. There is a symmetric obstruction theory for such pairs, given by combining the $R\text{Hom}_\perp$ perfect obstruction theory for (E, ϕ) of [TT1] with Joyce-Song's pairs theory. We speculate that the resulting invariants $P_{r,L,c_2}^\perp(n)$ can be written in terms of universal formulae in n with coefficients given by (and defining) Vafa-Witten invariants. When $H^{0,1}(S) = 0 = H^{0,2}(S)$ the formulae are the same as Joyce-Song's universal formulae for invariants defined by Behrend-weighted Euler characteristics:

$$(1.7) \quad P_{r,L,c_2}^\perp(n) = \sum_{\substack{\ell \geq 1, (\alpha_i = \delta_i \alpha)_{i=1}^\ell: \\ \sum_{i=1}^\ell \delta_i = 1}} \frac{(-1)^\ell}{\ell!} \prod_{i=1}^\ell (-1)^{\chi(\alpha_i(n))} \chi(\alpha_i(n)) \text{VW}_{\alpha_i}(S).$$

(Here $\alpha = (r, c_1(L), c_2)$; see (2.3) for a full explanation of the notation.) When either of $H^{0,1}(S)$ or $H^{0,2}(S)$ is nonzero, the formulae simplify and we conjecture only the first term in the sum survives:

$$(1.8) \quad P_{r,L,c_2}^\perp(n) = (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \text{VW}_{r,L,c_2}(S).$$

These formulae recover the invariants VW (1.1) when stability and semistability coincide, and give the “right” answer in many other calculations.

1.4. The two components. When stability and semistability coincide, there are two types of connected component of $\mathcal{N}_{r,L,c_2}^{\mathbb{C}^*}$ which contribute to the invariants VW and vw:

(1.9) \mathcal{M}_{r,L,c_2} , the moduli space of semistable sheaves of fixed determinant L on S . These are considered as Higgs pairs by setting $\phi = 0$, or, equivalently, as torsion sheaves on X by pushing forward from S .

(1.10) We let \mathcal{M}_2 denote the union of *all other* components of $(\mathcal{N}_{r,L,c_2}^\perp)^{\mathbb{C}^*}$, i.e. those for which ϕ is nilpotent but nonzero. They can be described in terms of *nested Hilbert schemes of S* ; see [GSY1, GSY2] and [TT1, Section 8].

When there are strictly semistable Higgs pairs, \mathcal{M}_2 (1.10) may not be closed but might touch \mathcal{M}_{r,L,c_2} (1.9).

The literature has hitherto focussed on the first of these components. By [TT1, Section 7.1], when there are no strictly semistables, its contribution to \mathbf{VW}_{r,L,c_2} is the (integer!) virtual signed Euler characteristic

$$(1.11) \quad \int_{[\mathcal{M}_{r,L,c_2}]^{\text{vir}}} c_{\text{vd}}(E^\bullet) \in \mathbb{Z},$$

of the instanton moduli space \mathcal{M}_{r,L,c_2} . Here $E^\bullet \rightarrow \mathbb{L}_{\mathcal{M}_{r,L,c_2}}$ is the natural obstruction theory, or virtual cotangent bundle, of \mathcal{M}_{r,L,c_2} , and we take its “virtually top” Chern class c_{vd} , where

$$\text{vd} = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S)$$

is its virtual dimension. This is the Ciocan-Fontanine-Kapranov/Fantechi-Göttsche signed Euler characteristic of \mathcal{M}_L studied in [JT].

Similarly \mathcal{M}_{r,L,c_2} ’s contribution to \mathbf{vw}_{r,L,c_2} is just its signed topological Euler characteristic

$$(1.12) \quad (-1)^{\text{vd}(\mathcal{M}_{r,L,c_2})} e(\mathcal{M}_{r,L,c_2})$$

by the dimension reduction result of [BBS, Da] described in [JT, Section 5].³

There is a large literature computing these topological Euler characteristics (1.12) in examples. Almost all of them satisfy some kind of positive curvature condition (such as $K_S \leq 0$) to enforce a “*vanishing theorem*” [VW, Section 2.4] to ensure that

- $\mathcal{M}_2 = \emptyset$, and
- \mathcal{M}_{r,L,c_2} is smooth.

Furthermore they work in situations where

- \mathcal{M}_{r,L,c_2} contains only *stable* sheaves.

The first condition ensures that $\pm e(\mathcal{M}_{r,L,c_2})$ is the *only* contribution to the Vafa-Witten invariants \mathbf{vw} , while the second implies that $\pm e(\mathcal{M}_{r,L,c_2})$ also equals the virtual invariant (1.11) in these examples, so $\mathbf{vw}_{r,L,c_2} = \mathbf{VW}_{r,L,c_2}$ and both are integers.

There are three references we know of with computations of Vafa-Witten invariants when no vanishing theorem holds. Noncompact surfaces given by line bundles over curves are studied in [AOSV, Section 3], while [GGP, Section 3] studies Vafa-Witten invariants via TQFT for 4-manifolds made by gluing. Most recently the preprint [GK] computes the contribution (1.11) of \mathcal{M}_{r,L,c_2} to \mathbf{VW}_{r,L,c_2} on general type surfaces. However none of these references calculate on the “other” component \mathcal{M}_2 (1.10), nor with strictly semistables.

³Vafa and Witten would not have the sign $(-1)^{\text{vd}}$ due to different orientation conventions. They identify the tangent and cotangent bundles of \mathcal{M}_{r,L,c_2} using a Riemannian metric. This is natural from the real point of view, but changes the natural complex orientation (that we use) by $(-1)^{\dim_{\mathbb{C}}}$.

Even when all Higgs pairs are stable, the contribution of \mathcal{M}_2 to VW_{r,L,c_2} can be *rational*, while vw_{r,L,c_2} is still an integer. In particular, once a vanishing theorem does not hold, vw and VW can differ. In [TT1] we calculated the contribution of \mathcal{M}_2 in examples with only stable Higgs pairs, and in this paper we also calculate with strictly semistables (at which point both invariants vw , VW become rational numbers.) Such calculations suggest which invariant is “correct” for physics, as we now explain.

1.5. Modularity. Vafa and Witten use “S-duality” to predict that, for fixed rank r and determinant L , the generating function

$$Z_r(S) := q^{-s} \sum_{n \in \mathbb{Z}} \text{VW}_{r,L,n}(S) q^n$$

should be a modular form for some appropriate shift s . Their examples indicated that one should take $s = e(S)/12$, and that the result is a modular form of weight $w/2 = -e(S)/2$. There is a large literature confirming these conjectures for the contribution of \mathcal{M}_{r,L,c_2} (1.9) in examples where a vanishing theorem holds and semistability implies stability.

In [TT1] we made some computations of the contributions of \mathcal{M}_2 to VW on surfaces with $K_S > 0$ satisfying some mild conditions (for instance to ensure that semistability implies stability). Generic quintic surfaces, and K3 surfaces blown up in a point are examples. For rank $r = 2$, determinant $L = K_S$ and arbitrary c_2 there is a natural series of Hilbert schemes $S^{[n]}$ amongst the components of \mathcal{M}_2 (1.10) and we managed to sum the generating series of their contributions to VW into a closed form.

The result was an *algebraic function* of q , rather than a modular form. Conversely in Section 6 of this paper we show the same calculation of the Kai localised invariants vw (1.5) gives a modular form up to a factor of $(1-q)^{e(S)}$. Incredibly,⁴ however, this is **not** an indication that vw is preferable to VW . On adding the contributions of other components of \mathcal{M}_2 the generating series of VW invariants gives precisely the modular form predicted by Vafa and Witten in low degrees [TT1, Section 8]. The invariants are rational numbers depending on $c_1(K_S)^2$ and $c_2(S)$, whereas the vw integers depend only on the Euler characteristic $c_2(S)$ and give the “wrong” answers.

The examples in [TT1] involved only stable Higgs pairs. In the presence of strictly semistables it is tempting to use the Joyce-Song/Kontsevich-Soibelman theory, which uses Behrend functions. And the resulting rational numbers vw indeed appear to give the right invariants on K3 surfaces, as we show in Section 5. This fact also misled us, since vw gives the wrong answers when $K_S > 0$. The resolution appears to be that we should expect⁵ $\text{vw} = \text{VW}$ on surfaces with $K_S \leq 0$, and indeed we do computations showing that VW *also* gives the correct answers on K3 surfaces in some examples with strictly semistables.

⁴This issue misled us for some time.

⁵In Theorem 7.9 we prove this expectation when $\deg K_S < 0$.

Threefold S-duality. Nonetheless we expect there to be a role for *both* definitions VW and vw in the S-duality conjecture for *threefolds* [MSW, GaSY, DeM, dB⁺]. Namely the DT theory of sheaves supported on surfaces in a compact Calabi-Yau 3-fold X should also have modular generating series. These invariants might be expected to be localised to a sum of invariants local to surfaces in X . One could then take either form of localisation — virtual or Behrend — to recover either type of Vafa-Witten invariant for these surfaces. For each individual surface they might give different answers, but their *sum* over all surfaces in X should give the same modular form when X is compact.

And there has been compelling work showing that the Behrend approach is compatible with threefold S-duality. In particular Yukinobu Toda has found many modular generating series from weighted Euler characteristics (see for instance his blow-up formula [To2, Theorem 4.3] and calculations on local \mathbb{P}^2 [To3] and local K3 [To1]). His work, and that of Manschot et al [Ma, ABMP], also shows that wall-crossing transformations (on generating functions of DT invariants counting two dimensional sheaves on Calabi-Yau 3-folds such as $X = K_S$) preserve the modularity predicted by S-duality when one uses weighted Euler characteristics.

Finally, Greg Moore and Sergei Gukov explained to us another advantage of the invariant vw over VW. Physics predicts that Vafa-Witten theory admits a categorification or refinement, given by a topological twist of maximally supersymmetric 5d super Yang-Mills theory. With current mathematical technology, this can only be done to the Kai-localised invariant vw, not the virtually localised VW. So it seems natural to expect both invariants to play useful roles.

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Notation This paper is currently written in a less formal style than [TT1]; it is more of a description of a future research programme than Theorem-Proof mathematics. Some is conjectural and some proofs are only sketched.

We pass backwards and forwards through the spectral construction without comment. See [TT1] for a detailed review; in particular the equivalence of abelian categories

$$\mathrm{Higgs}_{K_S}(S) \cong \mathrm{Coh}_c(X)$$

between K_S -Higgs pairs (E, ϕ) on S and compactly supported coherent sheaves on X [TT1, Proposition 2.2]. This equates Gieseker (semi)stability of the pair (E, ϕ) with respect to $\mathcal{O}_S(1)$ with Gieseker (semi)stability of the sheaf \mathcal{E}_ϕ with respect to $\mathcal{O}_X(1) := \pi^* \mathcal{O}_S(1)$.

For rank r and second Chern class c_2 we use the notation

$$\mathcal{M}_{r,L,c_2} \subset \mathcal{M}_{r,c_1,c_2}$$

for the moduli space of semistable sheaves of determinant L (respectively first Chern class $c_1 = c_1(L)$) on S . We reserve \mathcal{M}_S for the moduli *stack* of all coherent sheaves on S . Similarly

$$\mathcal{N}_{r,L,c_2}^\perp \subset \mathcal{N}_{r,L,c_2} \subset \mathcal{N}_{r,c_1,c_2}$$

denote the moduli spaces of semistable Higgs pairs with Chern classes $r, c_1 = c_1(L), c_2$ and $\det E = L$ in the first and second cases and also $\text{tr } \phi = 0$ in the first case. From the second and third of these spaces we will define invariants $\text{vw}, \widehat{\text{vw}}$ by Kai localisation, while from the first and third we will define $\text{VW}, \widehat{\text{VW}}$ by virtual localisation. In the presence of semistables we will work with corresponding spaces of Joyce-Song pairs, denoted by

$$\mathcal{P}_{r,L,c_2}^\perp \subset \mathcal{P}_{r,L,c_2} \subset \mathcal{P}_{r,c_1,c_2},$$

with corresponding invariants P_{r,L,c_2}^\perp (defined by virtual localisation in Section 7) and $P_{r,L,c_2}, \tilde{P}_{r,c_1,c_2}$ (defined by weighted Euler characteristics in Sections 3.1 and 4.1 respectively.)

2. SEMISTABLE SHEAVES AND JOYCE-SONG THEORY

2.1. Gieseker (semi)stability. We say a Higgs pair (E, ϕ) on $(S, \mathcal{O}_S(1))$ is Gieseker stable if and only if

$$(2.1) \quad p_F(n) := \frac{\chi(F(n))}{\text{rank}(F)} < \frac{\chi(E(n))}{\text{rank}(E)} =: p_E(n) \quad \text{for } n \gg 0,$$

for every ϕ -invariant proper subsheaf $F \subset E$ on S . Replacing $<$ by \leq defines Gieseker semistability. We call $p_E(n)$ the reduced Hilbert polynomial of E .

Gieseker (semi)stability of (E, ϕ) is equivalent to Gieseker (semi)stability of the spectral sheaf \mathcal{E}_ϕ with respect to $\mathcal{O}_X(1) = \pi^* \mathcal{O}_S(1)$. This is defined by the inequality

$$\frac{\chi(\mathcal{F}(n))}{r(\mathcal{F})} < \frac{\chi(\mathcal{E}_\phi(n))}{r(E)} \quad \text{for } n \gg 0,$$

for all subsheaves $\mathcal{F} \subset \mathcal{E}$. Here $r(\mathcal{E}) := \text{rank}(\pi_* \mathcal{E}) \int_S c_1(\mathcal{O}_S(1))^2$ denotes the leading coefficient of the Hilbert polynomial $\chi(\mathcal{E}(n))$ on X .

We fix the Chern classes

$$\text{rank}(E) = r, \quad c_1(E) = c_1, \quad c_2(E) = k$$

of our Higgs pairs (E, ϕ) on S . Equivalently, via the spectral construction, we consider compactly supported torsion sheaves on X with rank 0 and

$$\begin{aligned}
 c_1 &= r[S], \\
 (2.2) \quad c_2 &= -\iota_* \left(c_1 + \frac{r(r+1)}{2} c_1(S) \right), \\
 c_3 &= \iota_* \left(c_1^2 - 2k + (r+1)c_1 \cdot c_1(S) + \frac{r(r+1)(r+2)}{6} c_1(S)^2 \right)
 \end{aligned}$$

in $H_c^*(X, \mathbb{Z})$. Here $\iota: S \hookrightarrow X$ is the zero section and $[S]$ its Poincaré dual.

We combine these classes into the charge

$$(2.3) \quad \alpha = (r, c_1, k) \in H^{\text{ev}}(S).$$

If we fix $c_1 = 0$ we often denote this by $\alpha = (r, k) \in H^0(S) \oplus H^4(S)$. The Euler pairing on X of two charges α, β is defined to be

$$\chi(\alpha, \beta) := \chi_X(\mathcal{E}, \mathcal{F}) = \sum (-1)^i \text{ext}_X^i(\mathcal{E}, \mathcal{F}),$$

where \mathcal{E}, \mathcal{F} are any two torsion sheaves on X whose pushdown to S have charges α, β respectively. (Note that we confusingly work on X while expressing charges in terms of data on S .) This pairing is skew-symmetric; in particular for any charge α ,

$$\chi(\alpha, \alpha) \equiv 0.$$

Similarly we have the Hilbert polynomial and reduced Hilbert polynomial of the class α ,

$$\chi(\alpha(n)) := \chi_X(\mathcal{E}(n)) \quad \text{and} \quad p_\alpha(n) := \frac{\chi(\alpha(n))}{r}.$$

We also assume that the polarisation $\mathcal{O}_S(1)$ is *generic* so that

$$(2.4) \quad p_\beta(n) = \text{const} \cdot p_\alpha(n) \implies \beta = \text{const} \cdot \alpha.$$

(This assumption restricts the possible sheaves that destabilise \mathcal{E} , and so simplifies the formula (3.2) below. It is purely for simplicity, we can ignore it at the expense of using more complicated formulae from [JS].)

There is a quasi-projective moduli space parameterising S-equivalence classes of Gieseker semistable sheaves on X with fixed charge α . Its \mathbb{C}^* -fixed locus is projective, so we would like to define invariants by localising to it. This is no problem when stability and semistability coincide, but in general points of the moduli space represent an entire S-equivalence class of semistable sheaves (rather than a single sheaf), so it is not immediately clear how to count them correctly.

2.2. Hall algebra. Joyce-Song and Kontsevich-Soibelman therefore replace the moduli space by the moduli *stack* of semistable sheaves, and use its Behrend function to define generalised DT invariants which are *rational* numbers in general.

We describe some of this theory using the formalism of Joyce-Song's non-compact book [JS]. This requires two assumptions that do not always hold when $X = K_S$:

- X should be “*compactly embeddable*” [JS, Section 6.7], and
- $H^1(\mathcal{O}_X) = 0$.

Both conditions are only used to ensure that moduli of sheaves on X are locally analytically critical loci. The first allows them — when working with moduli of sheaves — to pretend that X is compact, while the second makes the line bundle $\mathcal{O}_X(n)$ spherical. Apply the spherical twist about it (for $n \ll 0$) therefore turns moduli of sheaves into moduli of bundles. The latter can be studied by analytic gauge theoretic methods to prove they are locally analytical critical loci (of the holomorphic Chern-Simons function). This is used to prove identities about Behrend functions.

Team Joyce has since proved that moduli stacks of sheaves on Calabi-Yau 3-folds are always locally algebraic critical loci [BBBJ], so we can ignore the above conditions.

Joyce [Jo2] defines a Ringel-Hall algebra. He starts with the \mathbb{Q} -vector space on generators given by (isomorphism classes of) morphisms of stacks from algebraic stacks of finite type over \mathbb{C} with affine stabilisers to the stack of objects of $\text{Coh}_c(X)$. He then quotients out by the scissor relations for closed substacks. We are interested in the elements

$$1_{\mathcal{N}_\alpha^{ss}} : \mathcal{N}_\alpha^{ss} \hookrightarrow \text{Higgs}_{K_S}(S) \cong \text{Coh}_c(X),$$

where \mathcal{N}_α^{ss} is the *stack* of Gieseker semistable Higgs pairs (E, ϕ) of class α on X , and $1_{\mathcal{N}_\alpha^{ss}}$ is its inclusion into the stack of all Higgs pairs on S .

To handle the stabilisers of strictly semistable sheaves, Joyce replaces these indicator stack functions by their “logarithm”,

$$(2.5) \quad \epsilon(\alpha) := \sum_{\substack{\ell \geq 1, (\alpha_i)_{i=1}^\ell : p_{\alpha_i} = p_\alpha \ \forall i \\ \text{and } \sum_{i=1}^\ell \alpha_i = \alpha}} \frac{(-1)^\ell}{\ell} 1_{\mathcal{N}_{\alpha_1}^{ss}} * \cdots * 1_{\mathcal{N}_{\alpha_\ell}^{ss}}.$$

In this finite sum $*$ denotes the Hall algebra product. At the level of individual objects, the product of (the indicator functions of) (E, ϕ) and (F, ψ) is (the indicator function of) the stack of all extensions between them,

$$\frac{\text{Ext}^1(\mathcal{F}_\psi, \mathcal{E}_\phi)}{\text{Aut}(\mathcal{E}_\phi) \times \text{Aut}(\mathcal{F}_\psi) \times \text{Hom}(\mathcal{F}_\psi, \mathcal{E}_\phi)}.$$

More generally it is defined via the stack \mathfrak{Ert} of all short exact sequences

$$(2.6) \quad 0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

in $\text{Coh}_c(X)$, with its morphisms $\pi_1, \pi, \pi_2 : \mathfrak{Ert} \rightarrow \text{Coh}_c(X)$ taking the extension to $\mathcal{E}_1, \mathcal{E}, \mathcal{E}_2$ respectively. This defines the universal case, which is the Hall algebra product of $\text{Coh}_c(X)$ with itself:

$$1_{\text{Coh}_c(X)} * 1_{\text{Coh}_c(X)} = \left(\mathfrak{Ert} \xrightarrow{\pi} \text{Coh}_c(X) \right).$$

Other products are defined by fibre product with this: given two stack functions $U, V \rightarrow \mathrm{Coh}_c(X)$ we define $U * V \rightarrow \mathrm{Coh}_c(X)$ by the Cartesian square

$$(2.7) \quad \begin{array}{ccc} U * V & \longrightarrow & \mathfrak{Ert} \xrightarrow{\pi} \mathrm{Coh}(X/S) \\ \downarrow & & \downarrow \pi_1 \times \pi_2 \\ U \times V & \longrightarrow & \mathrm{Coh}_c(X) \times \mathrm{Coh}_c(X). \end{array}$$

A deep result of Joyce [Jo3, Theorem 8.7] is that the logarithm (2.5) lies in the set of *virtually indecomposable stack functions with algebra stabilisers*,

$$\epsilon(\alpha) \in \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathrm{Coh}_c(X), e, \mathbb{Q}).$$

By [JS, Proposition 3.4] it can thus be written as a \mathbb{Q} -linear combination of morphisms from stacks of the form (scheme) $\times B\mathbb{C}^*$, where $B\mathbb{C}^*$ is the quotient stack $(\mathrm{Spec} \mathbb{C})/\mathbb{C}^*$. This allows Joyce-Song [JS, Section 5.3] to take the Kai-weighted Euler characteristic of the stack $\epsilon(\alpha)$ after removing the $B\mathbb{C}^*$ factor (they prove this “integration map” factors through $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathrm{Coh}_c(X), e, \mathbb{Q})$). The weighting is the Behrend function χ^B on $\mathrm{Coh}_c(X)$. That is, writing

$$(2.8) \quad \epsilon(\alpha) = \sum_i c_i (f_i: Z_i \times B\mathbb{C}^* \longrightarrow \mathrm{Coh}_c(X)),$$

where the Z_i are *schemes*, [JS, Equation 3.22] defines generalised DT invariants by

$$(2.9) \quad JS_\alpha(X) = \sum_i c_i e(Z_i, f_i^* \chi^B) \in \mathbb{Q}.$$

We can localise this invariant. The action of \mathbb{C}^* on X/S induces an action on the stack of torsion sheaves by pullback. Similarly pulling back the universal extension over $\mathfrak{Ert} \times X$ we find that if U, V are stacks with \mathbb{C}^* actions and *equivariant* morphisms to $\mathrm{Coh}_c(X)$, then the diagram (2.7) and their Hall algebra product $U * V$ inherit natural \mathbb{C}^* actions. Applied inductively to the $1_{\mathcal{N}_\alpha^{ss}}$ and their Hall algebra products, we find that $\epsilon(\alpha)$ (2.5) carries a \mathbb{C}^* action covering that on $\mathrm{Coh}_c(X)$.

We claim moreover that in its decomposition (2.8), the pieces Z_i can be taken to be \mathbb{C}^* -equivariant. This follows from the proof of the decomposition in [Jo1], where the key is to use Kresch’s stratification of finite type algebraic stacks with affine geometric stabilisers into global quotient stacks [Kr, Proposition 3.5.9]. This can be done \mathbb{C}^* -invariantly, as can the other constructions in [Jo1, Proposition 5.21].

Since the Behrend function is \mathbb{C}^* -invariant, non-fixed \mathbb{C}^* orbits on the Z_i have vanishing weighted Euler characteristic. As a result (2.9) localises to the fixed locus,

$$(2.10) \quad JS_\alpha(X) = JS_\alpha^{\mathbb{C}^*}(X) := \sum_i c_i e\left(Z_i^{\mathbb{C}^*}, f_i^* \chi^B|_{Z_i^{\mathbb{C}^*}}\right) \in \mathbb{Q}.$$

We use these localised invariants of X to define certain $U(r)$ Vafa-Witten invariants of S .

3. $U(r)$ $\widetilde{\text{vw}}$ INVARIANT

Definition 3.1. We define a $U(r)$ Vafa-Witten invariant of S by

$$\widetilde{\text{vw}}_{r,c_1,c_2}(S) := JS_{(r,c_1,c_2)}^{\mathbb{C}^*}(X) \in \mathbb{Q}.$$

Definition 3.1 generalises the invariant (1.5) from the Introduction to the semistable case. It is only useful when $h^1(\mathcal{O}_S) = 0$ because otherwise the action of $\text{Jac}(S)$ on $\text{Coh}_c(X)$ by tensoring forces it to vanish. In Section 4 we define a more useful $SU(r)$ Vafa-Witten invariant $\text{vw}_{r,c_1,c_2}(S)$.

While calculating with (2.10) directly is difficult, Joyce and Song prove their invariants may be written more simply in terms of certain *Joyce-Song stable pairs*. We review these next.

3.1. Joyce-Song pairs. Fixing $n \gg 0$, a Joyce-Song pair (\mathcal{E}, s) consists of

- a compactly supported coherent sheaf \mathcal{E} on X , and
- a section $s \in H^0(\mathcal{E}(n))$.

We say that the Joyce-Song pair (\mathcal{E}, s) is *stable* if and only if

- \mathcal{E} is Gieseker semistable with respect to $\mathcal{O}_X(1)$, and
- if $\mathcal{F} \subset \mathcal{E}$ is a proper subsheaf which destabilises \mathcal{E} , then s does *not* factor through $\mathcal{F}(n) \subset \mathcal{E}(n)$.

There is no notion of semistability; when X is compact the moduli space $\mathcal{P} = \mathcal{P}_{r,c_1,k}(X)$ of stable Joyce-Song pairs is already a projective scheme. As a moduli space of complexes $I^\bullet := \{\mathcal{O}_X(-n) \rightarrow \mathcal{E}\}$ on a Calabi-Yau 3-fold, \mathcal{P} carries a symmetric perfect obstruction theory. It may be noncompact, but we can still define *integer* invariants by

$$\tilde{P}_{r,c_1,k}(n) := e(\mathcal{P}_{r,c_1,k}, \chi^B)$$

and localise them as in (1.3):

$$\tilde{P}_{r,c_1,k}(n) = \tilde{P}_{r,c_1,k}^{\mathbb{C}^*}(n) := e\left(\mathcal{P}_{r,c_1,k}^{\mathbb{C}^*}, \chi_{\mathcal{P}_{r,c_1,k}}^B \big|_{\mathcal{P}_{r,c_1,k}^{\mathbb{C}^*}}\right).$$

Then for generic polarisation (2.4) Joyce-Song's invariants $JS_\alpha(X) = JS_{(r,c_1,k)}(X) \in \mathbb{Q}$ satisfy the following identities [JS, Theorem 5.27],

$$(3.2) \quad \tilde{P}_{r,c_1,k}(n) = \sum_{\substack{\ell \geq 1, (\alpha_i = \delta_i \alpha)_{i=1}^\ell \\ \sum_{i=1}^\ell \delta_i = 1}} \frac{(-1)^\ell}{\ell!} \prod_{i=1}^\ell (-1)^{\chi(\alpha_i(n))} \chi(\alpha_i(n)) JS_{\alpha_i}(X).$$

These equations have been simplified by (2.4). If we work with non-generic $\mathcal{O}_S(1)$, they should be replaced by the full equations of [JS, Theorem 5.27]. They uniquely determine the $JS_\alpha(X) = \widetilde{\text{vw}}_\alpha(S)$, and can be used to define them.

When semistability = stability for the sheaves \mathcal{E} , the moduli space $\mathcal{P}_{r,c_1,k}$ is a $\mathbb{P}^{\chi(\alpha(n))-1}$ -bundle over the moduli space \mathcal{N} of torsion sheaves \mathcal{E} . The Behrend function of \mathcal{P} is pulled back and multiplied by the sign $(-1)^{\chi(\alpha(n))-1}$ so taking Euler characteristics gives

$$\tilde{P}_{r,c_1,k}(n) = (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \widetilde{\text{vw}}_{r,c_1,k}(S),$$

which is the first term $\ell = 1$ of (3.2). In this case Definition 3.1 agrees with the naive definition (1.5).

More generally the $\ell > 1$ terms in (3.2) give rational corrections from semistable sheaves \mathcal{E} . For instance, when $r = 2$ and $c_1 = 0$ the two cases (depending on the parity of k) are

$$\tilde{P}_{2,2k+1}(n) = (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \widetilde{\text{vw}}_{2,2k+1}(S)$$

when $\alpha = (2, 2k + 1)$, and

$$\tilde{P}_{X,2,2k}(n) = (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \widetilde{\text{vw}}_{2,2k}(S) + \frac{1}{2} \chi\left(\frac{\alpha}{2}(n)\right)^2 \widetilde{\text{vw}}_{1,k}(S)^2$$

when $\alpha = (2, 2k)$.

4. $SU(r)$ vw INVARIANT

Definition 3.1 gives $\widetilde{\text{vw}} \equiv 0$ when $h^{0,1}(S) > 0$ because of the action of $\text{Jac}(S)$ on $\text{Coh}_c(X)$ by tensoring. So we modify Joyce-Song's theory by fixing the determinant of our sheaves $E = \pi_* \mathcal{E}$. (Even when $h^{0,1}(S) = 0$ the resulting $SU(r)$ Vafa-Witten invariant vw is slightly different from the $U(r)$ invariant $\widetilde{\text{vw}}$ because we remove the deformations $H^0(K_S)$ of the trace of the Higgs field.)

We fix a line bundle $L \in \text{Pic}(S)$ and use the map

$$\text{Coh}_c(X) \xrightarrow{\det \circ \pi_*} \text{Pic}(S).$$

We denote the fibre over L by

$$\text{Coh}_c(X)^L := (\det \circ \pi_*)^{-1}(L).$$

Then, given any stack function $F := (f: U \rightarrow \text{Coh}_c(X))$ we can define its fibre over $L \in \text{Pic}(S)$ to be

$$F^L := (f: U \times_{\text{Coh}_c(X)} \text{Coh}_c(X)^L \rightarrow \text{Coh}_c(X)).$$

This is $1_{\text{Coh}_c(X)^L} \cdot F$, where \cdot is the ordinary (not Hall!) product described in [JS, Definition 2.7].

Applied to Joyce's logarithm (2.8) we get its fixed determinant analogue

$$\epsilon(\alpha)^L := \sum_i c_i \left(f_i: (Z_i \times_{\text{Coh}_c(X)} \text{Coh}_c(X)^L) / \mathbb{C}^* \rightarrow \text{Coh}_c(X) \right).$$

Applying Joyce-Song's integration map to this gives a generalised fixed-determinant DT invariant

$$JS_\alpha^L(X) := \sum_i c_i e(Z_i \times_{\text{Coh}_c(X)} \text{Coh}_c(X)^L, f_i^* \chi^B).$$

As usual, in practice we compute this by localising to \mathbb{C}^* -fixed points (and using Joyce-Song pairs in the next Section). As in (2.10) the \mathbb{C}^* action on X covering the identity on S induces \mathbb{C}^* actions on $\mathrm{Coh}_c(X)^L$ and $\epsilon(\alpha)^L$. As before we can therefore also take the Z_i to carry equivariant \mathbb{C}^* actions, so that

$$JS_\alpha^L(X) = JS_\alpha^{L, \mathbb{C}^*}(X) := \sum_i c_i e\left(Z_i^{\mathbb{C}^*} \times_{\mathrm{Coh}_c(X)} \mathrm{Coh}_c(X)^L, f_i^* \chi^B\right).$$

Definition 4.1. *The $SU(r)$ Vafa-Witten invariant is*

$$\mathbf{vw}_{r,L,k}(S) := (-1)^{h^0(K_S)} JS_{(r,k)}^L(X) \in \mathbb{Q}.$$

Here we have inserted the sign to account for the fact that we did not restrict our sheaves \mathcal{E} to have centre of mass 0 on each fibre of $X \rightarrow S$ (equivalently, we did not insist that $\mathrm{tr} \phi = 0$) in the construction of JS^L . In the $SU(r)$ moduli space this condition should be enforced, and its product with $H^0(K_S)$ (which translates torsion sheaves up the K_S fibres) gives the moduli space we have used. This only affects the Behrend function, and so the weighted Euler characteristic, by the sign $(-1)^{\dim H^0(K_S)}$.

When $h^{0,1}(S) = 0$ and $\mathcal{O}_S(1)$ is generic (2.4) this means we modify the pairs theory only by a sign, and (3.2) becomes

$$\tilde{P}_{r,c_1,k}(n) = \sum_{\substack{\ell \geq 1, (\alpha_i = \delta_i \alpha)_{i=1}^\ell: \\ \sum_{i=1}^\ell \delta_i = 1}} \frac{(-1)^\ell}{\ell!} \prod_{i=1}^\ell (-1)^{\chi(\alpha_i(n)) + h^0(K_S)} \chi(\alpha_i(n)) \mathbf{vw}_{\alpha_i}(S).$$

For $h^{0,1}(S) > 0$ we have to modify the pairs theory more significantly.

4.1. Joyce-Song pairs. We sketch how the Joyce-Song pairs theory gets modified in this fixed-determinant setting. We use the notation of [JS, Chapter 13], most of which goes through with only minor modification. We fix $n \gg 0$ and use the same auxiliary categories \mathcal{B}_{p_α} (whose objects are sheaves \mathcal{E} with reduced Hilbert polynomial a multiple of p_α , plus a vector space V and a linear map $V \rightarrow H^0(\mathcal{E}(n))$), and the same Euler forms $\bar{\chi}$ thereon. Everything is unchanged up until subsection 13.5, where Joyce-Song apply their integration map (weighted Euler characteristic) to their stack functions $\bar{\epsilon}_{(\alpha,1)}$ of Equations (13.25) or (13.26). We instead apply their integration map to their fixed determinant analogues.

That is, there is a forgetful map from the stack of objects of \mathcal{B}_{p_α} to the stack of objects of $\mathrm{Coh}_c(X)$, remembering only the sheaf \mathcal{E} . Thus, in their notation, we fix $L \in \mathrm{Pic}(S)$ and define

$$\bar{\epsilon}_{(\alpha,1)}^L := \bar{\epsilon}_{(\alpha,1)} \times_{\mathrm{Coh}_c(X)} \mathrm{Coh}_c(X)^L.$$

This is a virtual indecomposable because $\bar{e}_{(\alpha,1)}$ is. Applying their integration map $\tilde{\Psi}^{\mathcal{B}_{p\alpha}}$ to it gives the fixed-determinant analogue of their count of Joyce-Song pairs

$$P_{r,L,k}(n) := e\left(\mathcal{P}_{r,L,k}, \chi_{\mathcal{P}_{r,L,k}}^B\right).$$

Here $\mathcal{P}_{r,L,k}$ is the moduli space of Joyce-Song pairs (\mathcal{E}, s) with $\det \pi_* \mathcal{E} \cong L$ and charge $\alpha = (r, c_1(L), k) \in H^{\text{ev}}(S)$. As usual the invariant is calculated in practice by localisation:

$$P_{r,L,k}(n) = P_{r,L,k}^{\mathbb{C}^*}(n) := e\left((\mathcal{P}_{r,L,k})^{\mathbb{C}^*}, \chi^B|_{(\mathcal{P}_{r,L,k})^{\mathbb{C}^*}}\right).$$

Then applying $\tilde{\Psi}^{\mathcal{B}_{p\alpha}}((\cdot)^L)$ to Joyce-Song's equation (13.26) for $\bar{e}_{(\alpha,1)}$ gives the fixed-determinant analogue of the formula (3.2). We claim it is the following (much simpler!) formula when $\mathcal{O}_S(1)$ is generic (2.4).

Proposition 4.3. *When $h^{0,1}(S) > 0$, the invariant $P_{r,L,k}^{\mathbb{C}^*}(n)$ determines the $SU(r)$ Vafa-Witten invariant by*

$$(4.4) \quad P_{r,L,k}(n) = (-1)^{h^0(K_S)} (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \text{vw}_{r,L,k}(S).$$

Proof. We let $\alpha = (r, c_1 = c_1(L), k)$ and use (2.4) so that the only splittings of $\alpha = \sum_i \alpha_i$ into pieces of the same reduced Hilbert polynomial are of the form $\alpha_i = \delta_i \alpha$ with $\sum_i \delta_i = 1$.

This simplifies [JS, Equation 13.26]. The first ($l = 1$) term of the $(\cdot)^L$ piece gives what we want $P_{r,L,k}(n)$ to be:

$$(4.5) \quad \begin{aligned} -\tilde{\Psi}^{\mathcal{B}_{p\alpha}}\left([\bar{e}_{(0,1)}, \bar{e}_{(\alpha,0)}]^L\right) &= -\bar{\chi}((0,1), (\alpha,0)) \tilde{\Psi}^{\mathcal{B}_{p\alpha}}(\bar{e}_{(0,1)}) \tilde{\Psi}^{\mathcal{B}_{p\alpha}}((\bar{e}_{(\alpha,0)})^L) \\ &= (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) JS_{(r,k)}^L(X) \end{aligned}$$

by [JS, Proposition 3.13 and Equation 13.30].

We will show that the other terms contribute zero by induction on l . The base case is the second ($l = 2$) term in [JS, Equation 13.26], which contributes

$$(4.6) \quad \frac{1}{2} \tilde{\Psi}^{\mathcal{B}_{p\alpha}}\left([\bar{e}_{(0,1)}, \bar{e}_{(\alpha_1,0)}], \bar{e}_{(\alpha_2,0)}]^L\right),$$

where $\alpha_i = \delta_i \alpha$ for some δ_i with $\delta_1 + \delta_2 = 1$. We evaluate this by first pushing down first to $\text{Jac}(S)$ via the determinant of the sheaves parameterised by $\bar{e}_{(\alpha_1,0)}$, i.e. via

$$\det \circ \pi_* : \text{Coh}_c(X)_{\alpha_1} \longrightarrow \text{Pic}_{\delta_1 c_1}(S),$$

before then pushing down to a point. That is, over $M \in \text{Jac}(S)$ we compute

$$(4.7) \quad \frac{1}{2} \tilde{\Psi}^{\mathcal{B}_{p\alpha}}\left[[\bar{e}_{(0,1)}, \bar{e}_{(\alpha_1,0)}]^M, (\bar{e}_{(\alpha_2,0)})^{M^{-1} \otimes L}\right]$$

— the contribution of extensions (in both directions) between objects of the first stack (with determinant M) and objects of the second with determinant $M^{-1} \otimes L \in \text{Pic}_{\delta_2 c_1}(S)$. The result is a constructible function of M whose Euler characteristic we take over $\text{Pic}_{\delta_1 c_1}(S) \ni M$ to calculate (4.6).

But by [JS, Proposition 3.13 and Equation 13.30], (4.7) is

$$\frac{1}{2}\bar{\chi}((\alpha_1, 1), (\alpha_2, 0))\tilde{\Psi}^{\mathcal{B}_{p\alpha}}[\bar{\epsilon}_{(0,1)}, \bar{\epsilon}_{(\alpha_1,0)}]^M\tilde{\Psi}^{\mathcal{B}_{p\alpha}}((\bar{\epsilon}_{(\alpha_2,0)})^{M^{-1}\otimes L}).$$

We have already seen in (4.5) that the first $\tilde{\Psi}^{\mathcal{B}_{p\alpha}}$ term is independent of M . We claim that so is the second. Therefore the pushdown is a *constant* constructible function on $\text{Pic}_{\delta_1 c_1}(S)$. Since $h^{0,1}(S) > 0$ this has Euler characteristic zero, and (4.6) indeed vanishes.

To prove the claim we note that the stack \mathfrak{Ert} carries an action of $\text{Jac}(S)$ given by tensoring all 3 terms of (2.6) by any $M \in \text{Jac}(S)$. Under the projection $\pi_1 \times \pi_2$ it covers the diagonal $\text{Jac}(S)$ action on $\text{Coh}_c(X) \times \text{Coh}_c(X)$, and under the projection π it covers the usual $\text{Jac}(S)$ action on $\text{Coh}_c(X)$.

Applying this in the diagram (2.7) we find that given any two stacks U, V with $\text{Jac}(S)$ actions and *equivariant* morphisms $U, V \rightarrow \text{Coh}_c(X)$, their Hall algebra product $U * V \rightarrow \text{Coh}_c(X)$ inherits a $\text{Jac}(S)$ action. Applied inductively to the $1_{\mathcal{N}_{\alpha}^{ss}}$, we find that the $\epsilon(\alpha)$ and their Hall algebra products all carry a $\text{Jac}(S)$ action covering that on $\text{Coh}_c(X)$. This latter action is an isomorphism, so preserves the Behrend function. Therefore the $\text{Jac}(S)$ action takes $(\bar{\epsilon}_{(\alpha_2,0)})^{L_1}$ isomorphically to $(\bar{\epsilon}_{(\alpha_2,0)})^{L_2}$ preserving the Behrend function, so their integrals $\Psi^{\mathcal{B}_{p\alpha}}$ are the same.

The other terms $l \geq 3$ vanish for similar reasons. By induction they are of the form (a constant times)

$$(4.8) \quad \tilde{\Psi}^{\mathcal{B}_{p\alpha}}([F, \bar{\epsilon}_{(\alpha_l,0)}]^L)$$

where F is a stack function taking values in the objects of $\mathcal{B}_{p\alpha}$ with charge $((1 - \delta_l)\alpha, 1)$ whose pushdown to $\text{Pic}_{(1-\delta_l)c_1}(S)$,

$$(4.9) \quad M \mapsto \tilde{\Psi}^{\mathcal{B}_{p\alpha}}(F^M) \text{ is constant.}$$

Now (4.8) is the Euler characteristic of the constructible function

$$\begin{aligned} M &\mapsto \tilde{\Psi}^{\mathcal{B}_{p\alpha}}[F^M, (\bar{\epsilon}_{(\alpha_l,0)})^{M^{-1}\otimes L}] \\ &= \bar{\chi}(((1 - \delta_l)\alpha, 1), (\alpha_l, 0))\tilde{\Psi}^{\mathcal{B}_{p\alpha}}(F^M)\tilde{\Psi}^{\mathcal{B}_{p\alpha}}((\bar{\epsilon}_{(\alpha_l,0)})^{M^{-1}\otimes L}), \end{aligned}$$

by [JS, Proposition 3.13 and Equation 13.30]. By (4.9) this is also constant on $\text{Pic}_{(1-\delta_l)c_1}(S)$. Thus its Euler characteristic vanishes. \square

5. K3 SURFACES

We first need a foundational result: that \mathbb{C}^* -fixed Higgs pairs are in fact \mathbb{C}^* -equivariant. For simple (e.g. stable) pairs this is standard — one can apply [Ko, Proposition 4.4] to the sheaves \mathcal{E}_ϕ , for instance. For pairs with non-scalar automorphisms (e.g. strictly semistable pairs) we have to work a bit harder.

Proposition 5.1. *If (E, ϕ) is fixed by the \mathbb{C}^* action scaling ϕ then either $\phi = 0$ or E admits an algebraic \mathbb{C}^* action*

$$\Psi: \mathbb{C}^* \longrightarrow \text{Aut}(E)$$

such that $\Psi_t \circ \phi \circ \Psi_t^{-1} = t\phi$ for all $t \in \mathbb{C}^*$.

Proof. Since $(E, t\phi)$ must be isomorphic (E, ϕ) we get, for each $t \in \mathbb{C}^*$, an automorphism ψ_t of E which conjugates $t\phi$ into ϕ :

$$(5.2) \quad \psi_t \circ \phi \circ \psi_t^{-1} = t\phi.$$

We will show that the ψ_t may be chosen to define a \mathbb{C}^* action on E , i.e. such that $\psi_s \circ \psi_t = \psi_{st}$ for all $s, t \in \mathbb{C}^*$.

Fix t which generates a Zariski dense subset $\{t^n: n \in \mathbb{Z}\}$ of \mathbb{C}^* . Let $\lambda_j \in \mathbb{C}$ be the eigenvalues of ψ_t ; these are constant since S is compact. Let $V_{\lambda_j} = \ker(\psi_t - \lambda_j)^N$, $N \gg 1$, be the generalised eigenspaces of E , so

$$(5.3) \quad E = \bigoplus_j V_{\lambda_j}.$$

Then (5.2) gives the identities

$$\begin{aligned} \psi_t \phi &= t\phi \psi_t \implies (\psi_t - \lambda t)\phi = t\phi(\psi_t - \lambda) \\ &\implies (\psi_t - \lambda t)^N \phi = t^N \phi(\psi_t - \lambda)^N. \end{aligned}$$

Applied to $v \in V_\lambda$ we get zero, which shows that $\phi v \in V_{t\lambda}$. Therefore ϕ acts blockwise on the decomposition (5.3) mapping any V_λ to $V_{t\lambda}$. Therefore, if we define the block diagonal operator

$$(5.4) \quad \tilde{\Psi}_t = \bigoplus_j \lambda_j \text{id}_{V_{\lambda_j}}$$

then this also satisfies $\tilde{\Psi}_t \circ \phi \circ \tilde{\Psi}_t^{-1} = t\phi$.

Say that $\lambda_i \sim \lambda_j$ whenever there is some $n \in \mathbb{Z}$ such that $\lambda_i = t^n \lambda_j$. Then we can partition the λ_j into equivalence classes, in each of which we can pick a representative λ_0 and write $\lambda_j = t^{\mu_j} \lambda_0$ for some integers μ_j . Therefore replacing (5.4) by

$$(5.5) \quad \Psi_t = \bigoplus_j t^{\mu_j} \text{id}_{V_{\lambda_j}},$$

this *also* satisfies $\Psi_t \circ \phi \circ \Psi_t^{-1} = t\phi$. We deduce that $\Psi_{t^n} \circ \phi \circ \Psi_{t^n}^{-1} = t^n \phi$ for every $n \in \mathbb{Z}$, so by Zariski denseness we conclude that

$$\Psi_s \circ \phi \circ \Psi_s^{-1} = s\phi \quad \forall s \in \mathbb{C}^*.$$

Thus (5.5) defines our required \mathbb{C}^* action on E . □

We now show how the vw theory works on K3 surfaces. We first illustrate the theory with rather explicit calculations in rank 2, before switching to more abstract results in general rank.

5.1. Rank 2. We consider semistable rank $r = 2$ Higgs sheaves (E, ϕ) on a fixed polarised K3 surface $(S, \mathcal{O}_S(1))$. We fix $\det E = \mathcal{O}_S$ and $\text{tr } \phi = 0$. We use \mathfrak{t} to denote the one dimensional \mathbb{C}^* representation of weight 1.

Proposition 5.6. *If (E, ϕ) is Gieseker semistable and \mathbb{C}^* -fixed, then E is itself Gieseker semistable. Moreover, either $\phi = 0$ or $c_2(E) = 2k$ is even and (up to an overall twist by some power of \mathfrak{t})*

$$(5.7) \quad E = I_Z \oplus I_Z \cdot \mathfrak{t}^{-1} \quad \text{and} \quad \phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for some length- k subscheme $Z \subset S$.

Proof. Let (E, ϕ) be \mathbb{C}^* -fixed. If $\phi = 0$ then E is a semistable \mathbb{C}^* -fixed sheaf and we are done. So we now assume that $\phi \neq 0$. Then by Proposition 5.1 the sheaf E carries a \mathbb{C}^* action acting with weight 1 on ϕ . Thus $E = \oplus_i E_i$ splits into weight spaces E_i on which $\lambda \in \mathbb{C}^*$ acts as λ^i .

Since ϕ decreases weight it maps the lowest weight torsion subsheaf to zero. This subsheaf is therefore ϕ -invariant, and so zero by semistability. Therefore each of the E_i are torsion-free, and so in particular have rank > 0 . Thus they have rank 1, and there are only two of them:

$$E = E_i \oplus E_j,$$

with $i > j$ without loss of generality. Since the Higgs field has weight 1, it takes weight k to weight $k - 1$. It is also nonzero, so we must have $j = i - 1$ and the only nonzero component of ϕ maps E_i to E_{i-1} .

Tensoring E by \mathfrak{t}^{-i} (i.e. multiplying the \mathbb{C}^* action on E by $\lambda^{-i} \cdot \text{id}_E$) we may assume without loss of generality that $i = 0$ and $j = -1$. Considering ϕ as a weight 0 element of $\text{Hom}(E, E) \otimes \mathfrak{t}$, we have

$$(5.8) \quad E = E_0 \oplus E_{-1} \quad \text{and} \quad \phi = \begin{pmatrix} 0 & 0 \\ \Phi & 0 \end{pmatrix} \quad \text{for some } \Phi: E_0 \longrightarrow E_{-1} \cdot \mathfrak{t}.$$

Therefore $E_{-1} \subset E$ is ϕ -invariant, so by the definition of Higgs semistability we get the inequality

$$\chi(E_0(n)) \geq \chi(E_{-1}(n)) \quad \forall n \gg 0.$$

Since both E_i are torsion free, Φ is an injection, implying the opposite inequality

$$\chi(E_0(n)) \leq \chi(E_{-1}(n)) \quad \forall n \gg 0.$$

Hence Φ is actually an isomorphism, and $E = E_0 \oplus E_0 \cdot \mathfrak{t}^{-1}$, which is semistable because E_0 is.

Finally, since E_0 is rank 1 torsion free with trivial determinant, it is an ideal sheaf I_Z , where Z has length $c_2(E)/2$. \square

Proposition 5.9. *The Behrend function $\chi^B \equiv -1$ on the stack of all rank 2 semistable Higgs pairs (E, ϕ) with $\det E \cong \mathcal{O}_S$ and $\text{tr } \phi = 0$ on a K3 surface.*

Remark 5.10. Here our stack has full stabiliser groups, i.e. even simple (e.g. stable) Higgs pairs have stabiliser \mathbb{C}^* . If we rigidify by removing this, we change χ^B to $+1$. If we do not fix $\text{tr } \phi = 0$ we change the sign again, since $h^0(K_S) = 1$. In particular, the forgetful map from the moduli space of stable Joyce-Song pairs \mathcal{P}_α in class α to the stack of semistable Higgs pairs (with no condition on $\text{tr } \phi$) is smooth of dimension $\chi(\alpha(n))$, so \mathcal{P}_α has Kai function $\equiv (-1)^{\chi(\alpha(n))}$.

Proof. When $\phi = 0$ this is a by now well-known result called “dimension reduction” for (-1) -shifted cotangent bundles: the Behrend function is $(-1)^{\text{vd}}$ on its zero section [BBS, Da], [JT, Section 5]. Here vd is the virtual dimension $\text{ext}^1(E, E)_0 - \text{ext}^2(E, E)_0 - \text{hom}(E, E) = 1 - \chi(E, E)$ of the moduli stack of sheaves E on S with fixed determinant.

Away from the zero section, we need an explicit local model for the moduli stack of objects of $\text{Coh}_c(X)$ near the pairs (5.7). On S , rather than X , the local model is given by [KaLe, Proposition 3.3]: near $I_Z \oplus I_Z$, the moduli stack \mathcal{M}_S of sheaves on S looks like the product of $\text{Ext}_S^1(I_Z, I_Z)$ with the quotient by GL_2 of the zero locus of the cup product map

$$(5.11) \quad \begin{array}{ccc} \mathfrak{sl}_2 \otimes \text{Ext}_S^1(I_Z, I_Z) & \longrightarrow & \mathfrak{sl}_2 \\ A & \longmapsto & A \cup A \end{array}$$

Here \cup denotes the Lie bracket on \mathfrak{sl}_2 tensored with the cup product

$$\text{Ext}_S^1(I_Z, I_Z) \otimes \text{Ext}_S^1(I_Z, I_Z) \longrightarrow \text{Ext}_S^2(I_Z, I_Z) \cong \mathbb{C},$$

and GL_2 acts by the adjoint action on the left and the identity on the right.

The Higgs pair (5.7) is a point of the (-1) -shifted cotangent bundle $T^*[-1]\mathcal{M}_S$ of \mathcal{M}_S , with $\phi \in \text{Hom}(E, E)_0 \cong \mathfrak{sl}_2$ a point of the fibre over $E = I_Z \oplus I_Z \in \mathcal{M}_S$. From (5.11) and the description of (-1) -shifted cotangent bundles [JT, Proposition 2.8] we find a local model for $T^*[-1]\mathcal{M}_S$ about (E, ϕ) . It is the product of $\text{Ext}_S^1(I_Z, I_Z)$ with the quotient by GL_2 of the critical locus of the function

$$(5.12) \quad \begin{array}{ccc} \mathfrak{sl}_2 \otimes \text{Ext}_S^1(I_Z, I_Z) \oplus \mathfrak{sl}_2 & \longrightarrow & \mathbb{C} \\ (A, \phi) & \longmapsto & \text{tr}(\phi(A \cup A)). \end{array}$$

To describe this critical locus, fix a symplectic basis e_i, f_i for $\text{Ext}_S^1(I_Z, I_Z)$. That is, $\text{tr}(e_i \cup e_j) = 0 = \text{tr}(f_i \cup f_j)$ and $\text{tr}(e_i \cup f_j) = \delta_{ij}$. Writing $A \in \mathfrak{sl}_2 \otimes \text{Ext}_S^1(I_Z, I_Z)$ as $\sum_i A_{e_i} \otimes e_i + \sum_i A_{f_i} \otimes f_i$ with $A_{e_i}, A_{f_i} \in \mathfrak{sl}_2$, the derivative of the function (5.12) down $(a \otimes e_i, 0)$ is therefore

$$2 \text{tr}(\phi[a, A_{f_i}]) = -2 \text{tr}(a[\phi, A_{f_i}]).$$

The vanishing of this for all $a \in \mathfrak{sl}_2$ is equivalent to the vanishing of $[\phi, A_{f_i}]$. Replacing e_i by f_i we similarly get the vanishing of $[\phi, A_{e_i}]$ for all i . We conclude that at a point with $\phi \neq 0$, each A_{e_i}, A_{f_i} is proportional to ϕ , i.e.

$$(5.13) \quad A \in \langle \phi \rangle \otimes \text{Ext}_S^1(I_Z, I_Z).$$

In turn this forces the derivative

$$\mathrm{tr}(\psi(A \cup A))$$

of (5.12) down $(0, \psi)$ to vanish, so (5.13) is precisely the critical locus. (With a bit more care, writing out the equations via a basis for \mathfrak{sl}_2 , one can see that the scheme structure of the critical locus is the reduced one on the locus (5.13).)

In particular we see that for $\phi \neq 0$ the critical locus is smooth — it is an $\mathrm{Ext}_S^1(I_Z, I_Z)$ -bundle over $\mathfrak{sl}_2 \setminus \{0\}$. Multiplying by $\mathrm{Ext}_S^1(I_Z, I_Z)$ we get a smooth odd dimensional space whose Behrend function is therefore -1 . Dividing by GL_2 , which is even-dimensional, does not change this. \square

c_2 odd. When $c_2(E)$ is *odd*, by Proposition 5.6 the Higgs field vanishes, the sheaf E is *stable*, and the moduli space $\mathcal{N}_{2, c_2}^\perp$ is just the moduli space of instantons on S (pushed forward to X). This was observed in [VW] as a case where their vanishing theorem holds. In particular the moduli space is smooth, projective, hyperkähler, has $\chi^B \equiv 1$ and is deformation equivalent to

$$(5.14) \quad \mathrm{Hilb}^{2c_2(E)-3}(S).$$

So we can evaluate the contribution of odd c_2 to the generating function using Göttsche's formula

$$\sum_n q^n e(\mathrm{Hilb}^n S) = \left(\prod_{k=1}^{\infty} \frac{1}{(1 - q^k)} \right)^{e(S)} = q \eta(q)^{-24}.$$

The result is

$$(5.15) \quad \sum_{c_2 \text{ odd}} q^{c_2} e(\mathrm{Hilb}^{2c_2(E)-3} S) = \frac{1}{4} q^2 \left(\eta(q^{1/2})^{-24} + \eta(-q^{1/2})^{-24} - \eta((-q)^{1/2})^{-24} - \eta(-(-q)^{1/2})^{-24} \right).$$

c_2 even. For $c_2(E)$ even, however, no such vanishing result holds and we have to deal with strictly semistable Higgs pairs. We take a Joycean approach, and compare the result to the predictions of Vafa and Witten. By enforcing modularity of the final result, they conjectured that the generating function of invariants should be

$$(5.16) \quad \frac{1}{4} q^2 \eta(q^2)^{-24} + \frac{1}{2} q^2 \left(\eta(q^{1/2})^{-24} + \eta(-q^{1/2})^{-24} \right).$$

Here we have adjusted for the $\pm 1/|Z(G)| = \pm 1/2$ difference in our invariants, and omitted Vafa-Witten's shift q^{-2} . Putting these back in gives the modular form of [VW, Equation 4.17]. Taking only odd powers of q in (5.16) recovers (5.15).

In particular the prediction (5.16) starts with

$$\mathrm{vw}_{(2,0)}(S) = \frac{1}{4}, \quad \mathrm{vw}_{(2,1)}(S) = 0, \quad \mathrm{vw}_{(2,2)}(S) = \frac{24}{4} + 24 = 30,$$

which we shall now check explicitly as an illustration of the theory.

For the first we use

$$\mathbf{vw}_{(1,0)}(S) = 1,$$

counting the sheaf \mathcal{O}_S on X (rigid in the space of sheaves with fixed centre of mass 0 on the fibres of K_S), and

$$\begin{aligned} P_{2,0}(n) &= \frac{\chi(\mathcal{O}_S(n))(\chi(\mathcal{O}_S(n)) - 1)}{2} + (2\chi(\mathcal{O}_S(n)) - \chi(\mathcal{O}_S(n))) \\ (5.17) \quad &= \frac{1}{4}\chi(\alpha(n)) + \frac{1}{2}\chi\left(\frac{\alpha}{2}(n)\right)^2, \end{aligned}$$

where α is the class $(2,0)$ of $\mathcal{O}_S \oplus \mathcal{O}_S$. The first term is the Euler characteristic of the moduli space $\mathrm{Gr}(2, H^0(\mathcal{O}_S(n)))$ of stable Joyce-Song pairs with underlying sheaf $\mathcal{E} = \mathcal{O}_S \oplus \mathcal{O}_S$. For the second the \mathbb{C}^* -fixed sheaf is $\mathcal{E} = \mathcal{O}_{2S} := \mathcal{O}_X/I_{S \subset X}^2$ and the corresponding moduli space of stable Joyce-Song pairs is $\mathbb{P}(H^0(\mathcal{O}_S(2n)) \setminus H^0(\mathcal{O}_S(n))) / \mathbb{C}$, with Euler characteristic $2\chi(\mathcal{O}_S(n)) - \chi(\mathcal{O}_S(n))$. There is no additional sign, due to the (restriction to $\mathcal{P}_{2,0}(X)^{\mathbb{C}^*}$ of the) identity $\chi_{\mathcal{P}_{2,0}(X)}^B \equiv 1$ of Remark 5.10.

For a class α with divisibility 2 on a K3 surface S , (4.2) reads

$$(5.18) \quad P_\alpha(n) = \chi(\alpha(n))\mathbf{vw}_\alpha(S) + \frac{1}{2}\chi\left(\frac{\alpha}{2}(n)\right)^2 \mathbf{vw}_{\alpha/2}(S)^2$$

Comparing to (5.17) gives $\mathbf{vw}_{(2,0)}(S) = \frac{1}{4}$, as required.

The second prediction $\mathbf{vw}_{(2,1)}(X) = 0$ already follows from our analysis (5.14) of the odd c_2 case, of course.

So we are left with the third, $\mathbf{vw}_{(2,2)}(X) = 30$.

Lemma 5.19. *Any \mathbb{C}^* -invariant semistable sheaf \mathcal{E} on X of class $(2,2)$ is a strictly semistable extension of the form*

$$0 \longrightarrow \iota_* I_x \longrightarrow \mathcal{E} \longrightarrow \iota_* I_y \longrightarrow 0,$$

for points $x, y \in S$.

Proof. By Proposition 5.6, the sheaf $E = \pi_* \mathcal{E}$ is semistable on S ; thus $h^0(E) = 0$. But $\chi(E) = 2$, so

$$h^2(E) = \mathrm{hom}(E, \mathcal{O}_S) \geq 2.$$

So we may pick a nonzero map $\phi: E \rightarrow \mathcal{O}_S$. Its image is an ideal sheaf $I \subset \mathcal{O}_S$ which by the semistability of E can only have cokernel of dimension zero and length 0 or 1. It is therefore either \mathcal{O}_S or I_y for some point $y \in S$.

The kernel of ϕ is a rank 1 torsion free sheaf of trivial determinant and so is also an ideal sheaf I_Z . Since $c_2(E) = 2$ we find Z has length 2 or 1 in the two cases above. If the latter it takes the form I_x and we are done. If the former we get an exact sequence

$$(5.20) \quad 0 \longrightarrow I_Z \longrightarrow E \longrightarrow \mathcal{O}_S \longrightarrow 0$$

with Z of length 2. Pick a point $y \in Z$ such that $\text{Hom}(I_Z, I_y) = \mathbb{C}$. Since $H^1(I_y) = 0$, the long exact sequence of $\text{Hom}(\cdot, I_y)$ applied to (5.20) shows that $\text{Hom}(E, I_y) = \mathbb{C}$, and we can proceed as before. \square

So we can now classify \mathbb{C}^* -fixed stable Joyce-Song pairs with underlying semistable sheaf \mathcal{E} in class $\alpha = (2, 2)$.

- $\mathcal{E} = \iota_*(I_x \oplus I_y)$ with $x \neq y \in S$. The pairs moduli space is a $\mathbb{P}(H^0(I_x(n)) \times \mathbb{P}(H^0(I_y(n)))$ -bundle over $(S \times S \setminus \Delta_S)/\mathbb{Z}/2$ with Euler characteristic

$$\chi\left(\frac{\alpha}{2}(n)\right)^2 \frac{e(S)^2 - e(S)}{2}.$$

- $\mathcal{E} = \iota_*(I_x \oplus I_x)$ with $x \in S$. The moduli space of Joyce-Song pairs is a $\text{Gr}(2, H^0(I_x(n)))$ -bundle over S with Euler characteristic

$$\frac{1}{2} \chi\left(\frac{\alpha}{2}(n)\right) \left(\chi\left(\frac{\alpha}{2}(n)\right) - 1\right) e(S).$$

- \mathcal{E} is a *nontrivial extension* between I_x and itself, $x \in S$, classified by a point of

$$\mathbb{P}(\text{Ext}^1(I_x, I_x))^{\mathbb{C}^*} \cong \mathbb{P}(T_x X)^{\mathbb{C}^*} = \mathbb{P}(T_x S) \sqcup S,$$

where the very last term corresponds to the vertical K_S direction in $T_x X$. Then the pairs space is a $\mathbb{P}(H^0(\mathcal{E}(n)) \setminus H^0(I_x(n)))/\mathbb{C}$ -bundle over $\mathbb{P}(TS) \sqcup S$ with Euler characteristic

$$\left(\chi(\alpha(n)) - \chi\left(\frac{\alpha}{2}(n)\right)\right) 3e(S).$$

Adding it all up and using Remark 5.10 gives

$$P_{2,2}(n) = \frac{5}{4} \chi(\alpha(n)) e(S) + \frac{1}{2} \chi\left(\frac{\alpha}{2}(n)\right)^2 e(S)^2$$

Using

$$\mathbf{vw}_{\alpha/2}(S) = \mathbf{vw}_{(1,1)}(S) = e(S) = 24,$$

and comparing to (5.18) gives $\mathbf{vw}_{(2,2)}(S) = \frac{5}{4} e(S) = 30$, as required by modularity.

All \mathbf{c}_2 . For the general case we use the following conjecture of Toda [To1]:

$$(5.21) \quad JS_\alpha(X) = - \sum_{k \geq 1, k|\alpha} \frac{1}{k^2} e(\text{Hilb}^{1-\frac{1}{2}\chi_S(\frac{\alpha}{k}, \frac{\alpha}{k})} S)$$

for any class $\alpha \in H^*(S)$ and $X = S \times \mathbb{C}$. Here χ_S is the Mukai pairing *on* S instead of X (and is minus the pairing Toda uses) and we have added a sign because Toda uses bare Euler characteristics. He hints at the natural conjecture that χ^B should be ± 1 so that (5.21) gives the correct virtual answer; this is what we prove in rank 2 by Proposition 5.9.

In particular for $\alpha = (2, 2k)$ we get

$$(5.22) \quad \mathbf{vw}_{(2,2k)}(S) = e(\text{Hilb}^{4k-3} S) + \frac{1}{4} e(\text{Hilb}^k S),$$

while we already know from (5.14) that

$$(5.23) \quad \text{vw}_{(2,2k+1)}(S) = e(\text{Hilb}^{4k-1} S).$$

So assuming Toda's conjecture, the generating series is

$$\frac{1}{2}q^2(\eta(q^{1/2})^{-24} + \eta(-q^{1/2})^{-24}) + \frac{1}{4}\eta(q^2)^{-24},$$

the last term coming from the last term in (5.22), and the first term coming from the sum of the two remaining terms in (5.22, 5.23). But this is precisely the Vafa-Witten prediction (5.16).

5.2. All rank and all c_2 . We can do a similar analysis for rank r sheaves with trivial determinant and charges $\alpha = (r, n)$. Toda's conjecture (5.21) gives

$$\sum_n \text{vw}_{(r,n)}(S)q^n = \sum_{d|r} \frac{1}{d^2} \sum_{m \in \mathbb{Z}} e\left(\text{Hilb}^{\frac{r}{d}(m-\frac{r}{d})+1} S\right)q^{md},$$

where on the right we have summed over those $n = md$ divisible by d . Shifting m by the integer r/d and then swapping the roles of d and r/d and shows the generating series is

$$(5.24) \quad \sum_{d|r} \frac{d^2}{r^2} \sum_{m \in \mathbb{Z}} e\left(\text{Hilb}^{dm+1} S\right)q^{\frac{mr}{d}+r}.$$

To sum this we rewrite Göttsche's formula as

$$\sum_n e(\text{Hilb}^{n+1} S)q^n = \eta(q)^{-24}$$

and take only powers of q divisible by d on both sides to give

$$\sum_m e(\text{Hilb}^{md+1} S)q^{md} = \frac{1}{d} \sum_{j=0}^{d-1} \eta\left(e^{\frac{2\pi i j}{d}} q\right)^{-24}.$$

Substituting in (5.24) we find the generating series is

$$(5.25) \quad \sum_{d|r} \frac{d}{r^2} q^r \sum_{j=0}^{d-1} \eta\left(e^{\frac{2\pi i j}{d}} q^{\frac{r}{d^2}}\right)^{-24}.$$

When r is prime, so that d takes only the values 1 and r , this becomes

$$-\frac{1}{r^2} q^r \eta(q^r)^{-24} - \frac{1}{r} q^r \sum_{j=0}^{r-1} \eta\left(e^{\frac{2\pi i j}{r}} q^{1/r}\right)^{-24}.$$

In [VW, End of Section 4.1] Vafa and Witten made precisely this prediction, and asked for the extension to more general r , which is what (5.25) gives.

6. NESTED HILBERT SCHEMES

We note briefly one more Euler characteristic calculation, on a general surface S . For some time we found it suggestive that vw is the “correct” Vafa-Witten invariant, but as explained in Section 1.5 we no longer think this.

In [TT1, Section 8] we studied surfaces with $K_S > 0$ satisfying some mild conditions. When $c_1(E) = -c_1(S)$ the “other” part of the fixed locus \mathcal{M}_2 (1.10) turns out to be a union of nested Hilbert schemes $S^{[i,j]}$ of 0-dimensional subschemes $Z_i \subseteq Z_j \subset S$ of lengths $i \leq j$ respectively. We calculated the contribution of the series of components $S^{[0,j]} \cong S^{[j]}$ and found the generating series of invariants could be summed over all j to give an *algebraic function* instead of a modular form. (However we later found that in adding in some of the terms with nonzero i gave answers which agreed in low degree with modular forms predicted by Vafa and Witten.)

Calculating instead with topological Euler characteristics is much easier. (Ideally we would use Kai-weighted Euler characteristics, or prove an analogue Proposition 5.9 to show that the Behrend function is ± 1 , but we do not know this in general.) In fact we can sum over all i and j , and the result is very close to being modular.

By [St, Equation 1.128] the generating series of nested partitions is

$$\sum_{\mu \leq \lambda} q^{|\mu|+|\lambda|} = (1-q) \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right)^2.$$

Therefore by standard arguments we find the generating series of the bare Euler characteristic versions of the vw invariants is

$$\sum_{i \leq j} e(S^{[i,j]}) q^{i+j} = (1-q)^{e(S)} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n} \right)^{2e(S)}.$$

After multiplying by Vafa-Witten’s shift $q^{-s} = q^{-\frac{e(S)}{12}}$ this becomes

$$q^{-s} \sum_{i \leq j} e(S^{[i,j]}) q^{i+j} = (1-q)^{e(S)} \cdot \eta(q)^{-2e(S)}.$$

So *up to the rational function* $(1-q)^{e(S)}$ this is the weight $-e(S)$ modular form $\eta(q)^{-2e(S)}$.

We do not understand the significance of the rational factor, the modular form, nor the weight (which is twice the standard Vafa-Witten prediction). Note that a similar formula has recently been obtained for nested Hilbert schemes with 3 steps [Bo], relevant to $SU(3)$ Vafa-Witten theory. But all such formulae depend only on $c_2(S)$, whereas Vafa-Witten’s predictions also involve $c_1^2(S)$.

7. EXTENSION OF $SU(r)$ VW INVARIANT TO THE SEMISTABLE CASE

Finally we describe the virtual localisation version VW of the $SU(r)$ Vafa-Witten invariant vw described in Section 4.

Fix $n \gg 0$, $r > 0$ and $c_2 \in H^4(S)$, and recall the notion of a Joyce-Song pair (\mathcal{E}, s) from Section 3.1. So for us, \mathcal{E} is a pure dimension 2 Gieseker semistable sheaf on $X = K_S$ whose pushdown $E = \pi_* \mathcal{E}$ has rank r , determinant L and second Chern class c_2 . (Equivalently it is a semistable Higgs pair (E, ϕ) on S .) Then s is a section of $\mathcal{E}(n)$ which does not factor through any destabilising subsheaf. Let

$$\mathcal{P}_{r,L,c_2}^\perp \subset \mathcal{P}_{r,c_1,c_2}$$

denote the moduli space of Joyce-Song pairs with $\det E \cong L$ and $\text{tr } \phi = 0$.

In [JS, Chapter 12] Joyce-Song construct a symmetric perfect obstruction theory on \mathcal{P}_{r,c_1,c_2} which has tangent-obstruction complex (the shift by [1] of)

$$(7.1) \quad R\text{Hom}_X(I^\bullet, I^\bullet)_0$$

at the point $I^\bullet := \{\mathcal{O}_X(-n) \xrightarrow{s} \mathcal{E}\}$. This can be modified to give a symmetric perfect obstruction theory on $\mathcal{P}_{r,L,c_2}^\perp \subset \mathcal{P}_{r,c_1,c_2}$ by following [TT1, Section 5].⁶ That is, removing $H^1(\mathcal{O}_S) \oplus H^2(\mathcal{O}_S)[-1] \oplus H^0(K_S) \oplus H^1(K_S)[-1]$ from (7.1) gives the first term of the decomposition

$$\begin{aligned} R\text{Hom}_X(I^\bullet, I^\bullet)_0 &\cong R\text{Hom}_X(I^\bullet, I^\bullet)_\perp \\ &\quad \oplus H^*(\mathcal{O}_X) \oplus H^{\geq 1}(\mathcal{O}_S) \oplus H^{\leq 1}(K_S)[-1]. \end{aligned}$$

On the second line we have removed the deformation theory of $\det I^\bullet \cong \mathcal{O}_X$, of $\det \pi_* \mathcal{E} \cong \mathcal{O}_S$ and of $\text{tr } \phi \in \Gamma(K_S)$ respectively. Doing this in a family, shifting by [1] and dualising gives an obstruction theory on $\mathcal{P}_{r,L,c_2}^\perp$. Full details will appear in a future paper.

So we can now use virtual localisation to the fixed locus of the \mathbb{C}^* action scaling the K_S fibres of $X \rightarrow S$ (equivalently scaling the Higgs fields ϕ) to define invariants

$$P_{r,L,k}^\perp(n) := \int_{[(\mathcal{P}_{r,L,c_2}^\perp)^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})}.$$

We speculate that these satisfy similar⁷ identities to the Joyce-Song invariants $P_{r,c_1,k}(n)$, $P_{r,L,k}(n)$ defined by Kai localisation in Sections 3.1 and 4.1, with vw replaced by VW. We use the notation (2.3) from Section 2.2, and a generic polarisation $\mathcal{O}_S(1)$ (2.4).

⁶In [TT1, Section 5] we modified the obstruction theory for \mathcal{N} by removing $H^1(\mathcal{O}_S) \oplus H^2(\mathcal{O}_S)[-1] \oplus H^0(K_S) \oplus H^1(K_S)[-1]$ from $\tau^{[0,1]}(R\text{Hom}_X(\mathcal{E}, \mathcal{E})[1])$ to get an obstruction theory based on $R\text{Hom}_X(\mathcal{E}, \mathcal{E})_\perp[1]$ for \mathcal{N}_L^\perp .

⁷Similar, but nonetheless genuinely different when $h^{0,2}(S) > 0$.

Conjecture 7.2. *If $H^{0,1}(S) = 0 = H^{0,2}(S)$ there exist $\text{VW}_{\alpha_i}(S) \in \mathbb{Q}$ such that*

$$(7.3) \quad P_{\alpha}^{\perp}(n) = \sum_{\substack{\ell \geq 1, (\alpha_i = \delta_i \alpha)_{i=1}^{\ell} \\ \sum_{i=1}^{\ell} \delta_i = 1}} \frac{(-1)^{\ell}}{\ell!} \prod_{i=1}^{\ell} (-1)^{\chi(\alpha_i(n))} \chi(\alpha_i(n)) \text{VW}_{\alpha_i}(S)$$

for $n \gg 0$. When either of $H^{0,1}(S)$ or $H^{0,2}(S)$ is nonzero we take only the first term in the sum:

$$(7.4) \quad P_{r,L,c_2}^{\perp}(n) = (-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) \text{VW}_{r,L,c_2}(S).$$

The formula (7.4) is of course reminiscent of the formula (4.4) for invariants defined by weighted Euler characteristic when $h^{0,1}(S) > 0$, but is different for $h^{0,1}(S) = 0 < h^{2,0}(S)$. The motivation for dropping the other terms when $h^{0,1}(S) > 0$ or $h^{0,2}(S) > 0$ is that we think of them as enumerating the contributions of nontrivial direct sums of sheaves. When $h^{0,1}(S) > 0$ these come in families with a nontrivial $\text{Jac}(S)$ action, for instance with $M \in \text{Jac}(S)$ acting on $\mathcal{E}_1 \oplus \mathcal{E}_2$ by taking it to the sheaf $\mathcal{E}_1 \otimes M^{-r_2} \oplus \mathcal{E}_2 \otimes M^{r_1}$ with the same determinant. This defines a nowhere zero vector field, and so cosection of the obstruction sheaf, over these semistable loci. Similarly when $h^{0,2}(S) > 0$ these loci inherit extra trivial pieces in their trace-free obstruction spaces. So in both cases we expect their virtual contribution to be zero.

Since (3.2) is a wall crossing formula for invariants defined by weighted Euler characteristic it is natural to expect our conjecture to be proved by an extension of the different wall crossing formula of Kiem-Li [KL2]. Their work uses virtual localisation instead of the Behrend function, so should fit naturally with VW. This should also be used to clarify the invariance (expect in physics) of VW under changes in the polarisation $\mathcal{O}_S(1)$. We intend to return to this in future work.

We start by proving these conjectures — and showing they recover the invariants VW (1.1) — when stability and semistability coincide.

Proposition 7.5. *If all semistable sheaves in $\mathcal{N}_{r,L,c_2}^{\perp}$ are stable then Conjecture 7.2 is true with VW_{r,L,c_2} given by the definition (1.1) of [TT1].*

Proof. We sketch the proof using induction on the rank r of $\alpha = (r, c_1, k)$. We first claim that if there are no strictly semistables in class α then only the first term contributes to the sum (7.3). Indeed, if there was a nonzero contribution indexed by $\alpha_1, \dots, \alpha_{\ell}$ with $\ell > 1$ then the nonvanishing of the numbers $\text{VW}_{\alpha_i}(S)$ (which equal the numbers (1.1) by the induction hypothesis) would imply that the moduli spaces $\mathcal{N}_{\alpha_i}^{\perp}$ are nonempty. Picking an element \mathcal{E}_i of each defines a strictly semistable $\mathcal{E} := \mathcal{E}_1 \oplus \dots \oplus \mathcal{E}_{\ell}$ of $\mathcal{N}_{\alpha}^{\perp}$, a contradiction.

As there are no strictly semistables, and all stable sheaves are simple,

$$(7.6) \quad \text{the moduli space of pairs } \mathcal{P}_{\alpha}^{\perp} \text{ is a } \mathbb{P}^{\chi(\alpha(n))-1} \text{ bundle over } \mathcal{N}_{\alpha}^{\perp}.$$

Since in \mathbb{C}^* -equivariant K-theory, $I^\bullet = \mathcal{O}_X(-n) - \mathcal{E}$ we see the class of $R\mathrm{Hom}_X(I^\bullet, I^\bullet)[1]$ is

$$\begin{aligned} & -R\mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X) + R\mathrm{Hom}_X(\mathcal{O}_X(-n), \mathcal{E}) \\ & + R\mathrm{Hom}_X(\mathcal{E}, \mathcal{O}_X(-n)) - R\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}). \end{aligned}$$

Removing the trace and using Serre duality with $K_X \cong \mathcal{O}_X \otimes \mathfrak{t}^{-1}$ gives

$$(7.7) \quad -R\mathrm{Hom}_X(I^\bullet, I^\bullet)_0 = H^0(\mathcal{E}(n)) - H^0(\mathcal{E}(n))^* \otimes \mathfrak{t} - R\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}).$$

The piece $\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) \cong \mathbb{C} \cdot \mathrm{id}_{\mathcal{E}}$ injects into $H^0(\mathcal{E}(n))$ via the section s with quotient⁸ $T_{\mathbb{P}(H^0(\mathcal{E}(n)))}$. So removing $H^{\geq 1}(\mathcal{O}_S)[1] \oplus H^{\leq 1}(K_S)$ from (7.7) gives

$$(7.8) \quad -R\mathrm{Hom}_X(I^\bullet, I^\bullet)_\perp = T_{\mathcal{P}_\alpha^\perp/\mathcal{N}_\alpha^\perp} - T_{\mathcal{P}_\alpha^\perp/\mathcal{N}_\alpha^\perp}^* \otimes \mathfrak{t} - R\mathrm{Hom}_X(\mathcal{E}, \mathcal{E})_\perp.$$

The fixed locus $(\mathcal{P}_\alpha^\perp)^{\mathbb{C}^*}$ inherits a perfect obstruction theory from the fixed part of $R\mathrm{Hom}_X(I^\bullet, I^\bullet)_\perp[1]$. From (7.8) we see it is the natural one inherited from \mathcal{N}_α^\perp and the smooth bundle structure (7.6) (at least in equivariant K-theory, which is all we need).

In particular the virtual cycle of $(\mathcal{P}_\alpha^\perp)^{\mathbb{C}^*}$ is the pullback of that of $(\mathcal{N}_\alpha^\perp)^{\mathbb{C}^*}$, and

$$N_{(\mathcal{P}_\alpha^\perp)^{\mathbb{C}^*}}^{\mathrm{vir}} = N_{(\mathcal{N}_\alpha^\perp)^{\mathbb{C}^*}}^{\mathrm{vir}} - T_{\mathcal{P}_\alpha^\perp/\mathcal{N}_\alpha^\perp}^* \otimes \mathfrak{t}.$$

Therefore, by the projection formula, $P_\alpha^\perp(n)$ is

$$\int_{[(\mathcal{P}_\alpha^\perp)^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N_{(\mathcal{P}_\alpha^\perp)^{\mathbb{C}^*}}^{\mathrm{vir}})} = \int_{\mathbb{P}} e(T_{\mathbb{P}}^* \otimes \mathfrak{t}) \int_{[(\mathcal{N}_\alpha^\perp)^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N_{(\mathcal{N}_\alpha^\perp)^{\mathbb{C}^*}}^{\mathrm{vir}})},$$

where $\mathbb{P} = \mathbb{P}^{\chi(\alpha(n))^{-1}}$ is any fibre of (7.6). This is

$$(-1)^{\chi(\alpha(n))^{-1}} \chi(\alpha(n)) \mathrm{VW}_\alpha(S). \quad \square$$

7.1. $\mathbf{K_S} < \mathbf{0}$. We can also prove the conjecture when $\deg K_S < 0$. While there may be strictly semistables, we find the space of Joyce-Song pairs is still smooth.

Theorem 7.9. *Suppose $\deg K_S < 0$. Then Conjecture 7.2 is true and $\mathrm{VW}_\alpha = \mathrm{vw}_\alpha$.*

Proof. By [TT1, Proposition 7.6] the semistable Higgs pairs are all of the form $(E, 0)$ for some semistable sheaf E on S with $\mathrm{Hom}(E, E \otimes K_S) = 0$. By Serre duality, then,

$$(7.10) \quad \mathrm{Ext}_S^2(E, E) = 0 = \mathrm{Hom}_S(E, E \otimes K_S).$$

⁸If we worked properly over the family \mathcal{P}_α^\perp this would be obvious. \mathcal{P}_α^\perp is the projectivisation of the bundle (twisted, if the universal sheaf is twisted) over \mathcal{N}_α^\perp with fibre $H^0(\mathcal{E}(n))$ over \mathcal{E} . As such it carries an $\mathcal{O}(1)$ (twisted) line bundle whose tensor product with the pullback of the universal sheaf is a genuine sheaf carrying a universal map from $\mathcal{O}_X(-n)$ defining the universal complex I^\bullet . Thus $H^0(\mathcal{E}(n))$ carries this twist by $\mathcal{O}(1)$ over $\mathbb{P}(H^0(\mathcal{E}(n)))$, so the quotient by $\mathbb{C} \cdot s$ is indeed $T_{\mathbb{P}(H^0(\mathcal{E}(n)))}$.

We work with the corresponding sheaf $\mathcal{E} = \iota_* E$ on X . Since $L\iota^* \iota_* E \cong E \oplus E \otimes K_S^{-1}[1]$ we have

$$(7.11) \quad \mathrm{Ext}_X^*(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}_S^*(E, E) \oplus \mathrm{Ext}_S^{*-1}(E, E \otimes K_S).$$

Fix any Joyce-Song pair $I^\bullet = \{\mathcal{O}_X(-n) \xrightarrow{s} \mathcal{E}\}$. As in (7.7),

$$(7.12) \quad -R\mathrm{Hom}_X(I^\bullet, I^\bullet)_0 = H^0(\mathcal{E}(n)) - H^0(\mathcal{E}(n))^* \otimes \mathfrak{t} - R\mathrm{Hom}_X(\mathcal{E}, \mathcal{E})$$

at the level of \mathbb{C}^* -equivariant K-theory. Now multiplication by s embeds $\mathrm{Hom}(E, E)$ into $H^0(\mathcal{E}(n))$ (by stability, because Joyce-Song pairs have no automorphisms). So, using (7.11) and (7.10) and removing $H^{\geq 1}(\mathcal{O}_S)[1] \oplus H^{\leq 1}(K_S)$, we find

$$(7.13) \quad \begin{aligned} -R\mathrm{Hom}_X(I^\bullet, I^\bullet)_\perp &= H^0(\mathcal{E}(n))/\mathrm{aut}(\mathcal{E}) - (H^0(\mathcal{E}(n))/\mathrm{aut}(\mathcal{E}))^* \otimes \mathfrak{t} \\ &+ \mathrm{Ext}_S^1(E, E)_0 - \mathrm{Ext}_S^1(E, E \otimes K_S)_0 \otimes \mathfrak{t}. \end{aligned}$$

It is easy to see this identity holds not just at the level of K-theory but at the level of cohomology sheaves, with the two fixed terms describing the tangent space to \mathcal{P}^\perp at I^\bullet . In particular there are no fixed obstructions, so \mathcal{P}^\perp is smooth and its own virtual cycle, with the above showing that $N^{\mathrm{vir}} = T_{\mathcal{P}^\perp}^* \otimes \mathfrak{t}[-1]$. Therefore

$$P_\alpha^\perp(n) = \int_{\mathcal{P}^\perp} e(T_{\mathcal{P}^\perp}^* \otimes \mathfrak{t}) = (-1)^{\dim \mathcal{P}^\perp} e(\mathcal{P}^\perp).$$

But this is precisely the answer that the Behrend theory gives, since by smoothness $\chi_{\mathcal{P}^\perp}^B \equiv (-1)^{\dim \mathcal{P}^\perp}$. Since \mathbf{VW} and \mathbf{vw} are determined from these two sets of pair invariants by the same formulae, we deduce that $P_\alpha^\perp(n)$ satisfies the equations of Conjecture 7.2 with $\mathbf{VW} = \mathbf{vw}$. \square

7.2. $\mathbf{K}_S = \mathbf{0}$. Numerous calculations — one of which we give below — lead us to believe that $\mathbf{VW} = \mathbf{vw}$ on K3 surfaces, despite the different universal formulae (4.2, 7.4) that determine them from pairs invariants. Then Conjecture 7.2 would follow from the corresponding results for Joyce-Song invariants defined by weighted Euler characteristics (Proposition 4.3).

Rahul Pandharipande explained to us an idea for relating \mathbf{vw} and \mathbf{VW} on a K3 or abelian surface S . We outline this now, though there are currently too many technical issues to turn it into a proof that $\mathbf{vw} = \mathbf{VW}$.

Let E be an elliptic curve. There is a “*reduced*” version of DT theory on $S \times E$ [OP, Ob]. Since $S \times E$ is a compact Calabi-Yau it can be defined in two ways that ultimately give the same answer. We can quotient the moduli space of sheaves by the translation action of E and remove the Serre dual trivial quotient of the obstruction sheaf, producing a symmetric obstruction theory (closely related to our obstruction theory $R\mathrm{Hom}(\mathcal{E}, \mathcal{E})_\perp[1]$ of [TT1]) and so a virtual cycle. Or we can fix the centre of mass of the sheaf in the E directions⁹ and taking Kai-weighted Euler characteristics.

⁹Oberdieck and Pandharipande use an insertion.

Now degenerate E to a nodal rational elliptic curve. By Jun Li's relative theory, the invariants can then be expressed in terms of sheaves on (expanded degenerations of) $S \times \mathbb{P}^1$. To glue $S \times \{0\}$ to $S \times \{\infty\}$ we take the fibre product of the moduli space with itself over the moduli space of boundary values on S .

For the first reduced theory, we can now apply virtual localisation to the \mathbb{C}^* action on \mathbb{P}^1 . The contribution from $S \times \{0\}$ comes from sheaves supported in the bubbles $S \times \mathbb{C}$, and should be expressible in terms of the obstruction theory $R\mathrm{Hom}(\mathcal{E}, \mathcal{E})_{\perp}[1]$ of [TT1], and thus in terms of the invariants VW . Similarly for $S \times \{\infty\}$. (Sheaves with nontrivial support at both 0 and ∞ have an extra trivial piece of their obstruction theory and so do not contribute.)

Using the Behrend theory instead is currently more difficult. Maulik and the second author have a long-stalled project to express sheaf counting invariants on log Calabi-Yau 3-folds in terms of weighted Euler characteristics. Given a 3-fold X with anticanonical divisor D , there are relative invariants (defined using Jun Li's theory) counting sheaves on X whose restriction to D lies in a fixed complex Lagrangian (in the holomorphic symplectic moduli space of sheaves on the Calabi-Yau surface D). This theory should be expressible in terms of an alternating sum of weighted Euler characteristics of strata of the moduli space, where the stratification is by the number of blow ups the sheaf is supported on in the expanded degeneration.

Applying this to $S \times \mathbb{P}^1$ and $D = S \times \{0, \infty\}$ and the (Lagrangian) diagonal in the moduli space $\mathcal{M}_S \times \mathcal{M}_S$ of sheaves on D should give a description of the invariants of $S \times E$ in terms of the invariants vw . Putting everything together would relate vw and VW .

While turning this into a proof is currently out of reach we make do with a suggestive calculation of the virtual theory VW on a K3 surface (to compare with the calculations of the Kai theory vw on K3 in Section 5). We work in $SU(r)$ Vafa-Witten theory with \mathbb{C}^* -fixed Higgs pairs which have the least possible degeneracy in their Higgs field (i.e. it is a Jordan block). Equivalently we work with \mathbb{C}^* -fixed torsion sheaves on $X = K_S$ with the largest possible scheme-theoretic support: the r times thickening rS of the zero section $S \subset X$.

Semistable sheaves with support rS are all of the form

$$\mathcal{E} = (\pi^* I_Z) \otimes \mathcal{O}_{rS}$$

for some ideal sheaf $I_Z \in S^{[k]}$. Thus $E \cong I_Z \oplus I_Z \mathfrak{t}^{-1} \oplus \cdots \oplus I_Z \mathfrak{t}^{-(r-1)}$ with $\det E = \mathcal{O}_S$ and class $\alpha = (r, rk)$, and

$$H_X^0(\mathcal{E}(n)) \cong H_S^0(I_Z(n)) \otimes (\mathbb{C} \oplus \mathfrak{t}^{-1} \oplus \cdots \oplus \mathfrak{t}^{-(r-1)}).$$

Any \mathbb{C}^* -fixed section s defining a Joyce-Song stable pair must lie in the first summand by stability, so $(\mathcal{P}^\perp)^{\mathbb{C}^*}$ is a $\mathbb{P}(H^0(I_Z(n)))$ -bundle over $S^{[k]}$. As usual we let $I^\bullet = \{\mathcal{O}_X(-n) \xrightarrow{s} \mathcal{E}\}$. Then by a similar calculation to (7.12,

7.13) we find that in \mathbb{C}^* -equivariant K-theory,

$$-R\mathrm{Hom}_X(I^\bullet, I^\bullet)_\perp = \frac{H^0(\mathcal{E}(n))}{\mathrm{aut}(\mathcal{E})} - \left(\frac{H^0(\mathcal{E}(n))}{\mathrm{aut}(\mathcal{E})} \right)^* \otimes \mathfrak{t} + \mathrm{Ext}_X^1(\mathcal{E}, \mathcal{E})_\perp - \mathrm{Ext}_X^2(\mathcal{E}, \mathcal{E})_\perp.$$

Replacing \mathcal{E} by its natural resolution $\pi^*I_Z(-rS) \rightarrow \pi^*I_Z$ gives

$$\begin{aligned} R\mathrm{Hom}_X(\mathcal{E}, \mathcal{E}) &= R\mathrm{Hom}_S(I_Z, I_Z) \otimes (\mathbb{C} \oplus \mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)}) \\ &\quad \oplus R\mathrm{Hom}_S(I_Z, I_Z) \otimes (\mathfrak{t}^r \oplus \dots \oplus \mathfrak{t})[-1], \end{aligned}$$

so that

$$\mathrm{Ext}_X^1(\mathcal{E}, \mathcal{E})_\perp \cong T_Z S^{[k]} \otimes (\mathbb{C} \oplus \mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)}) \oplus (\mathfrak{t}^r \oplus \dots \oplus \mathfrak{t}^2),$$

while $\mathrm{Ext}_X^2(\mathcal{E}, \mathcal{E})_\perp \cong \mathrm{Ext}_X^1(\mathcal{E}, \mathcal{E})_\perp^* \otimes \mathfrak{t}$. Now $\mathrm{Aut}(\mathcal{E}) \cong \mathbb{C}^* \ltimes (\mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)})$ and

$$\frac{H^0(\mathcal{E}(n))}{\mathrm{aut}(\mathcal{E})} \cong \frac{H^0(I_Z(n))}{\langle s \rangle} \otimes (\mathbb{C} \oplus \mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)})$$

is $T_{\mathbb{P}(H^0(I_Z(n)))} \otimes (\mathbb{C} \oplus \mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)})$ when we work in a family and the twisting of the universal section is taken into account as in Footnote 8. Putting everything together, the pairs invariant $P_\alpha^\perp(n)$ is

$$\int_{(\mathcal{P}^\perp)^{\mathbb{C}^*}} \frac{e(T^*\mathfrak{t})e(T^*\mathfrak{t}^2) \dots e(T^*\mathfrak{t}^r)e(\mathfrak{t}^{-1} \oplus \dots \oplus \mathfrak{t}^{-(r-1)})}{e(T\mathfrak{t}^{-1})e(T\mathfrak{t}^{-2}) \dots e(T\mathfrak{t}^{-(r-1)})e(\mathfrak{t}^r \oplus \dots \oplus \mathfrak{t}^2)},$$

where $T = T_{(\mathcal{P}^\perp)^{\mathbb{C}^*}} = T_{S^{[k]}} + T_{(\mathcal{P}^\perp)^{\mathbb{C}^*}/S^{[k]}}$. Since $e(A^*)/e(A) = (-1)^{\mathrm{rank} A}$ some cancellation gives

$$\begin{aligned} \int_{(\mathcal{P}^\perp)^{\mathbb{C}^*}} (-1)^{p(r-1)} e(T^*\mathfrak{t}^r) \frac{(-t)(-2t) \dots (-(r-1)t)}{(2t) \dots (rt)} = \\ (-1)^{pr} e(\mathbb{P}(H^0(I_Z(n)))) e(S^{[k]}) \frac{(-1)^{r-1}}{r}, \end{aligned}$$

where $p = \dim(\mathcal{P}^\perp)^{\mathbb{C}^*} = \chi(I_Z(n)) - 1 + 2k$. Since $\chi(\alpha(n)) = r\chi(I_Z(n))$ the final result is

$$P_\alpha^\perp(n) = (-1)^{\chi(\alpha(n))-1} \frac{\chi(\alpha(n))}{r^2} e(S^{[k]}).$$

This satisfies Conjecture 7.2, contributing $e(S^{[k]})/r^2$ to the Vafa-Witten invariant $\mathrm{VW}_{r,rk}(S)$ and

$$\sum_{k=0}^{\infty} \frac{e(S^{[k]})}{r^2} q^{rk} = \frac{1}{r^2} q^r \eta(q^r)^{-24}$$

to its generating series.

This matches the second term of [VW, Equation 5.38] (with $g-1 := c_1^2(S) = 0$), and the first term ($d=1$) of the generating series (5.25) of vw invariants. However, the vw calculation was somewhat different, with contributions from different components giving different answers for the pairs

invariants, but then the different universal formulae (4.2, 7.4) these satisfy lead to the same Vafa-Witten invariants $\mathbf{vw} = \mathbf{VW}$.

This can be seen even in the simplest example with $k = 0$ and $r = 2$, so that $\alpha = (2, 0)$. Then the moduli space of \mathbb{C}^* -fixed Joyce-Song pairs has two components — one a space of sections of $\mathcal{O}_{2S}(n)$, and one a Grassmannian $\mathrm{Gr}(2, H^0(\mathcal{O}_S(n)))$ of pairs with underlying sheaf $\mathcal{O}_S^{\oplus 2}$. The above virtual localisation calculation gives

$$(7.14) \quad P_\alpha^\perp(n) = \frac{1}{4}(-1)^{\chi(\alpha(n))-1} \chi(\alpha(n)) = -\frac{1}{4} \chi(\alpha(n))$$

for the first component and *zero* for the second.¹⁰ By contrast, the topological Euler characteristic of the first and second components are

$$\chi(\alpha(n)) - \chi\left(\frac{\alpha}{2}(n)\right) \quad \text{and} \quad \frac{1}{2} \chi\left(\frac{\alpha}{2}(n)\right) \left(\chi\left(\frac{\alpha}{2}(n)\right) - 1\right)$$

respectively. ($\chi^B \equiv 1$ here, by Remark 5.10, so we can ignore the weighting.) These are very different from (7.14), as is their sum $P_\alpha(n)$. But putting $P_\alpha^\perp(n)$ into (4.2) and $P_\alpha(n)$ into (7.4) gives the same Vafa-Witten invariants $\mathbf{VW}_\alpha = \mathbf{vw}_\alpha$.

7.3. $\mathbf{K_S} > \mathbf{0}$. Finally we describe a rather trivial calculation on general type surfaces S . It is nonetheless suggestive that the result takes the correct form to satisfy our Conjecture.

Take a surface S with $h^{0,1}(S) = 0$ and $h^{0,2}(S) > 0$ and charge $\alpha = (2, 0, 0)$. There is a $\mathbb{P}(\Gamma(K_S)) \ni [\sigma]$ of strictly semistable \mathbb{C}^* -fixed trivial determinant trace-free Higgs pairs of the form

$$E = \mathcal{O}_S \oplus \mathcal{O}_S \mathfrak{t}^{-1}, \quad \phi = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}.$$

We use the same deformation theory of Higgs pairs as in [TT1, Section 8.1]. In the obvious notation, the map $[\cdot, \phi]: \mathrm{Hom}(E, E) \rightarrow \mathrm{Hom}(E, E \otimes K_S)$ is

$$\begin{pmatrix} \mathbb{C} & \mathfrak{t} \\ \mathfrak{t}^{-1} & \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \Gamma(K_S) \mathfrak{t} & \Gamma(K_S) \mathfrak{t}^2 \\ \Gamma(K_S) & \Gamma(K_S) \mathfrak{t} \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} b\sigma & 0 \\ (d-a)\sigma & -b\sigma \end{pmatrix}.$$

Passing to the corresponding trace-free groups gives a map with kernel $\mathrm{End}_0(E, \phi) = \mathfrak{t}^{-1}$ and cokernel

$$\mathrm{Ext}^1(\mathcal{E}_\phi, \mathcal{E}_\phi)_\perp = \Gamma(K_S|_C) \mathfrak{t} \oplus \Gamma(K_S) \mathfrak{t}^2 \oplus \Gamma(K_S|_C),$$

where $C \subset S$ is the divisor of σ . Done properly, over the family $\mathbb{P}(\Gamma(K_S))$ as in Footnote 8 with s a section of $K_S \boxtimes \mathcal{O}_{\mathbb{P}(\Gamma(K_S))}(1)$, this is

$$(7.15) \quad T_{\mathbb{P}(\Gamma(K_S))} \mathfrak{t} \oplus \Gamma(K_S) \otimes \mathcal{O}_{\mathbb{P}(\Gamma(K_S))}(1) \mathfrak{t}^2 \oplus T_{\mathbb{P}(\Gamma(K_S))}.$$

¹⁰An easy calculation shows its \mathbb{C}^* -fixed obstruction bundle is the bundle of trace-free endomorphisms of the universal subbundle on $\mathrm{Gr}(2, H^0(\mathcal{O}_S(n)))$. But this has $c_3 = 0$ so the localised virtual cycle vanishes.

Now consider the Joyce-Song pairs $I^\bullet = \{\mathcal{O}_X(-n) \xrightarrow{s} \mathcal{E}_\phi\}$. Their moduli space \mathcal{N}^\perp restricts over the \mathbb{C}^* -fixed moduli space $\mathbb{P}(\Gamma(K_S))$ to a bundle with fibre the quotient of

$$(7.16) \quad \mathbb{P}(H^0(\mathcal{O}_S(n)) \oplus H^0(\mathcal{O}_S(n))\mathfrak{t}^{-1}) \backslash \mathbb{P}(H^0(\mathcal{O}_S(n))\mathfrak{t}^{-1})$$

by the obvious action of \mathfrak{t}^{-1} . Just as in (7.8) we find that

$$\mathrm{Ext}^1(I^\bullet, I^\bullet)_\perp = T_{\mathcal{N}^\perp/\mathbb{P}(\Gamma(K_S))} \oplus \mathrm{Ext}^1(\mathcal{E}_\phi, \mathcal{E}_\phi)_\perp.$$

Combining this with (7.15) and (7.16), we find that at a \mathbb{C}^* -fixed Joyce-Song pair $s = (s_1, 0)$ the tangent space $\mathrm{Ext}^1(I^\bullet, I^\bullet)_\perp$ is

$$(7.17) \quad \begin{aligned} & T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))} \oplus T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}\mathfrak{t}^{-1} \\ & \oplus T_{\mathbb{P}(\Gamma(K_S))}\mathfrak{t} \oplus \Gamma(K_S) \otimes_{\mathbb{P}(\Gamma(K_S))}(1)\mathfrak{t}^2 \oplus T_{\mathbb{P}(\Gamma(K_S))}. \end{aligned}$$

The first and last terms gives the fixed tangent space expected. Recalling the Serre duality $\mathrm{Ext}^2(I^\bullet, I^\bullet)_\perp \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_\perp^* \mathfrak{t}$, we find the third term gives a fixed obstruction bundle $T_{\mathbb{P}(\Gamma(K_S))}^*$. The virtual cycle is therefore its Euler class

$$(7.18) \quad [(\mathcal{P}^\perp)^{\mathbb{C}^*}]^{\mathrm{vir}} = (-1)^{h^0(K_S)-1} h^0(K_S) [\mathbb{P}(H^0(\mathcal{O}_S(n)))],$$

where $\mathbb{P}(H^0(\mathcal{O}_S(n)))$ is the fibre over a point of $\mathbb{P}(\Gamma(K_S))$. On these fibres the moving part of (7.17) simplifies to

$$T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}\mathfrak{t}^{-1} \oplus \mathfrak{t}^{\oplus h^0(K_S)-1} \oplus (\mathfrak{t}^2)^{\oplus h^0(K_S)}.$$

Similarly the moving part of $\mathrm{Ext}^2(I^\bullet, I^\bullet)_\perp \cong \mathrm{Ext}^1(I^\bullet, I^\bullet)_\perp^* \mathfrak{t}$ is

$$T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}^* \mathfrak{t} \oplus T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}^* \mathfrak{t}^2 \oplus (\mathfrak{t}^{-1})^{\oplus h^0(K_S)} \oplus \mathfrak{t}^{\oplus h^0(K_S)-1}.$$

Together these give the virtual normal bundle, with equivariant Euler class

$$\begin{aligned} e(N^{\mathrm{vir}}) &= \frac{e(T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}\mathfrak{t}^{-1}) \cdot t^{h^0(K_S)-1} \cdot (2t)^{h^0(K_S)}}{e(T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}^* \mathfrak{t}) e(T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}^* \mathfrak{t}^2) \cdot (-t)^{h^0(K_S)} \cdot t^{h^0(K_S)-1}} \\ &= (-1)^{\chi(\mathcal{O}_S(n))-1+h^0(K_S)} 2^{h^0(K_S)} \frac{1}{e(T_{\mathbb{P}(H^0(\mathcal{O}_S(n)))}^* \mathfrak{t}^2)}. \end{aligned}$$

Integrating $1/e(N^{\mathrm{vir}})$ over the virtual cycle (7.18) therefore gives

$$P_\alpha^\perp(n) = -2^{-h^0(K_S)} h^0(K_S) \chi(\mathcal{O}_S(n)).$$

Since $\chi(\alpha(n)) = 2\chi(\mathcal{O}_S(n))$ we see that this fits our conjecture (7.4) perfectly with contribution

$$\mathrm{VW}_\alpha = \frac{h^0(K_S)}{2^{h^0(K_S)+1}}.$$

The other \mathbb{C}^* -fixed component contains only (Joyce-Song pairs for) the semistable bundle $\mathcal{O}_S^{\oplus 2}$ with $\phi = 0$, and contributes nothing due to the trivial $H^0(K_S)$ piece of the obstruction space Ext_\perp^2 . Therefore $P_\alpha^\perp(n)$ has no terms quadratic in $\chi(\alpha(n))$, as predicted by the conjecture.

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